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PROPOSITIONAL Q-LOGIC

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ABSTRACT. Topic of the paper is Q-logic – a logic of agency in its temporal and modal context. Q-logic may be considered as a basal logic of agency since the most important *stit*-operators discussed in the literature can be defined or axiomatized easily within its semantical and syntactical framework. Its basic agent dependent operator, the Q-operator (also known as Δ- or cstit-operator), which has been discussed independently by F. v. Kutschera and B. F. Chellas, is investigated here in respect of its relation to other temporal and modal operators. The main result of the paper, then, is a completeness result for a calculus of Q-logic with respect to a semantics defined on the tree-approach to agency as introduced and developed by, among others, F. v. Kutschera and N. D. Belnap.

KEY WORDS: agency, branching histories, completeness, stit

1. INTRODUCTION

The most promising framework for the explication of action-theoretical concepts is provided by treelike structures that allow modeling the alternatives that different agents have in some moment and also the various alternatives that a single agent has in a sequence of moments. Within these structures it is possible to represent situations in which an agent is able to do something at some moment, while at some later moment she or he is not: an agent may miss chances to realize a certain state of affairs. The tree-approach to agency aims to represent exactly such situations.

The first complete presentation of the ideas on which the tree-approach of agency is based can be found in Franz von Kutschera's 'Bewirken'. Since then, the approach has been developed mainly by Nuel D. Belnap, Brian F. Chellas, John F. Horty, and Michael Perloff (cf., for example, Belnap and Perloff (1990, 1992), Belnap (1991a, 1991b), Chellas (1992), and Horty and Belnap (1995)). Belnap et al. (2001) provides an excellent survey of the discussions, and Horty (2001) presents applications of the tree-approach in the field of ethical theory. Furthermore, Ming Xu has discussed so-called *stit*-logics, i.e., logical systems dealing with Belnap's and Perloff's concept 'seeing to it that' (cf. (1994a, 1994b, 1995b, 1998)). However, his investigations leave out the temporal context of action sen-

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tences since the languages of the logical systems considered by him lack any temporal connective.

In this paper we will investigate the Q-operator, which has been discussed independently in Kutschera (1986) and in Chellas (1992). Chellas uses the symbol Δ instead of Kutschera's Q, and in the literature this operator is also referred to as cstit. The Q-operator functions as a binary operator that assigns to each agent name a and each sentence φ the sentence $\mathsf{Q}(a, \varphi)$ that may be read as: *the agent a settles φ as true*, or: *a necessitates φ*, or more informatively: *the alternative that is presently and actually chosen by a guarantees that φ is true*. This operator is introduced here with respect to two semantical approaches, one based on T×W-frames (cf. Kutschera (1986)) and one defined with respect to synchronized trees (cf., for example, Belnap et al. (2001)). The technical connections between both frame types are briefly discussed in Section 2. Then, in Section 3 some notions from the tree-approach to agency are presented as far as they are needed in the following sections.

The formal language introduced in Section 4 is an extension of a standard language for combinations of tense and modality. Besides usual temporal operators, it includes a series of other operators to express

– *time-dependent necessity* (historical necessity): Thomason (1984) presents a survey;
– *truth in all histories at the instant at hand*: this operator has been discussed first by Maria Concetta Di Maio and Alberto Zanardo in (1994) and, since then, it has been used in several completeness results of logical systems combining tense and modality with respect to T×W-semantics (cf. Kutschera (1997), Maio and Zanardo (1998), Wölfl (1999a, 1999b));
– *truth in all histories* (*passing through the moment at hand*) *distinct from the history at hand*: 'difference'-operators of this kind have been investigated, for example, in de Rijke (1992) and in Gargov and Goranko (1993).

The calculus of Q-logic studied in Sections 5–7 includes a Gabbay-style rule that is used in the completeness proof for constructing a canonical model. Readers who are interested in just this technical result may skip the following three sections.

2. Synchronized Trees and T×W-Frames

One reason for explaining action-theoretical concepts within treelike structures is given by the assumption of indeterminism, i.e., the assumption that there are events that actually happen, but are not causally determined by any previously occurring event. Indeterminism is the conditio sine qua non for any concept of agency that takes the idea seriously that agents can choose between different alternatives concerning what to do: normally, we are not inclined to consider an agent's behavior as an action unless the agent can refrain from performing this behavior. Thus, we obtain the picture of a *closed causal past* – events that happened in the past cannot be undone – but an *open causal future* – by acting we can, at least partially, influence the future. The picture of 'closed causal past, but open causal future' is captured by the concept of *tree*. The various (full) branches of a tree represent the various histories in which the world can evolve. Furthermore, with regard to applications of trees in action theory it is reasonable to assume that all branches of the considered trees have a unique temporal order, i.e., that there is an isomorphism between the temporal evolutions of all histories. Thus, we will restrict our following considerations of trees to such *synchronized trees*:

DEFINITION 2.1 (Synchronized tree). A *synchronized tree* is an ordered triple $\mathcal{B} = \langle Mom, \prec, Inst\rangle$ consisting of a non-void set *Mom* (the set of *moments*), an irreflexive, transitive and linear-to-the-left relation $\prec$ on *Mom* (the relation of *earlier-than*) – the set of all maximal $\prec$-chains in *Mom* is denoted by *His* (the set of *histories*) – and a partition *Inst* of *Mom* (the set of *instants*) such that the following conditions are satisfied:

(a) For every $i \in Inst$ and every $h \in His$ there exists exactly one $m_{i,h} \in Mom$ with $m_{i,h} \in i \cap h$.
(b) For all $i, i' \in Inst$ and all $h, h' \in His$, from $m_{i,h} \prec m_{i',h}$ it follows that $m_{i,h'} \prec m_{i',h'}$.

A quite similar, but more coordinate-like concept is given by T×W-*frames*:

DEFINITION 2.2 (T×W-frame). A T×W-*frame* is an ordered quadruple $\mathcal{R} = \langle T, <, W, \sim\rangle$ consisting of a non-void set T of *time-points*, a linear order-relation $<$ on T, a non-void set W of *possible worlds*, and a map $\sim$ that assigns to each $t \in T$ an equivalence relation $\sim_t$ on W such that the *no-backward-branching condition* holds:

$$w \sim_t w' \text{ and } t' < t \Longrightarrow w \sim_{t'} w',$$

for all $t, t' \in T$ and all $w, w' \in W$.

Some further conditions for T×W-frames are worth to be mentioned:

Modal Connectivity: For all worlds $w, w' \in W$ there exists a time-point $t \in T$ with $w \sim_t w'$.

Modal Divergence: For distinct worlds $w, w' \in W$ there is a time-point $t \in T$ with $w \not\sim_t w'$.

Modal Completeness: Let $\omega : T \to W$ be a map such that, for all $t < t'$, it holds that $\omega(t) \sim_t \omega(t')$. Then there exists a world $w^\omega \in W$ with $\omega(t) \sim_t w^\omega$ for each $t \in T$.

The first of these conditions will be discussed at the end of Section 6, the second in Section 7, and the third is the subject of consideration in this section. In general, we will not assume any of these conditions.

There is a well-known correspondence between T×W-frames and so-called *generalized* synchronized trees. But, how do the concept of (genuine) synchronized tree and that of T×W-frame correspond to each other? Starting with a synchronized tree $\mathcal{B}$, we obtain a T×W-frame $\mathcal{R}^{\mathcal{B}}$ by the following settings:

$$\begin{aligned}
&T := \mathit{Inst},\\
&W := \mathit{His},\\
&i < i' :\Longleftrightarrow m_{i,h} \prec m_{i',h}, \quad \text{for some/all } h \in \mathit{His},\\
&h \sim_i h' :\Longleftrightarrow m_{i,h} = m_{i,h'}.
\end{aligned}$$

The frame so defined is modally complete. For provided $\omega : \mathit{Inst} \to \mathit{His}$ is a map such that, for all $i < i'$, it holds that $m_{i,\omega(i)} \prec m_{i',\omega(i')}$, the set $\{m_{i,\omega(i)} : i \in \mathit{Inst}\}$ is a maximal $\prec$-chain in *Mom*, i.e., this set is a history. As can be verified easily, $\mathcal{R}^{\mathcal{B}}$ is modally divergent. And, the frame $\mathcal{R}^{\mathcal{B}}$ is modally connected if each pair of histories of $\mathcal{B}$ overlap.

Conversely, let $\mathcal{R}$ be a modally divergent and modally complete T×W-frame. Define an equivalence relation $\sim$ on the set $T \times W$ by

$$(t, w) \sim (t', w') :\Longleftrightarrow t = t' \text{ and } w \sim_t w'.$$

Equivalence classes w.r.t. (with respect to) $\sim$ will be designated in the form $[t, w]_\sim$. We then set

$$\begin{aligned}
&\mathit{Mom} := (T \times W)/\sim,\\
&[t, w]_\sim \prec [t', w']_\sim :\Longleftrightarrow t < t' \text{ and } w \sim_t w',\\
&[t, w]_\sim \approx [t', w']_\sim :\Longleftrightarrow t = t',\\
&\mathit{Inst} := \mathit{Mom}/\approx.
\end{aligned}$$

It is an easy exercise to check that the triple $\mathcal{B}^{\mathcal{R}} = \langle \mathit{Mom}, \prec, \mathit{Inst}\rangle$ defines a synchronized tree. Note that each world w determines a maximal $\prec$-chain of moments, namely

$$h_w := \big\{[t, w]_\sim \in \mathit{Mom} : t \in T\big\},$$

the history of world w. From modal completeness of $\mathcal{R}$ it follows, then, that the map $W \to His$, $w \to h_w$, is surjective, i.e., *His* and $\{h_w : w \in W\}$ are identical. And, the map $w \mapsto h_w$ is injective if and only if $\mathcal{R}$ is modally divergent. Furthermore, there exists a one-to-one correspondence between *Inst* and T given by $[[t, w]_\sim]_\approx \mapsto t$. Hence we obtain a bijection between the sets $T \times W$ and $Inst \times His$, and thus there is also a bijection between $T \times W$ and $\{(m, h) \in Mom \times His : m \in h\}$.[1]

There is a situation in which modal completeness is satisfied almost trivially:

PROPOSITION 2.3. *Let* $\mathcal{R}$ *be a* T×W*-frame such that the set of time-points* T *of* $\mathcal{R}$ *has a maximal element* $t_{\max}$. *Then* $\mathcal{R}$ *is modally complete.*

Proof. Let $\omega : T \to W$ be a map such that, for all $t, t' \in T$ with $t < t'$, $\omega(t) \sim_t \omega(t')$. By defining $w^\omega := \omega(t_{\max})$, we trivially obtain a world as wanted. □

It is exactly this fact that will be used in Section 7 to establish a completeness result of Q-logic with respect to a semantical approach based on synchronized trees with ending time. As far as I know, modal completeness itself is not axiomatized yet[2] and hence this condition will be neglected in the completeness proof presented here.[3]

3. Momentary Alternatives of Agents

After this brief survey on basic frame types and their mutual dependencies we will now discuss momentary alternatives of agents. At first, we introduce this concept with respect to synchronized trees:

DEFINITION 3.1 (Tree-based agent-frame). A *tree-based agent-frame* is an ordered triple $\mathcal{C} = \langle \mathcal{B}, Ag, Ch \rangle$, where $\mathcal{B}$ is a synchronized tree, Ag is a non-void set (the set of *agents* of $\mathcal{C}$), and Ch is a map that assigns to each agent $a \in Ag$ and each moment m of $\mathcal{B}$ a partition $Ch_a(m)$ of the set

$$His_m := \{h \in His : m \in h\}$$

such that the following conditions are met:

(a) If $h \in X \in Ch_a(m)$ and if h and h' are *undivided* at m (i.e., there is a moment $m' \in h \cap h'$ with $m \prec m'$), then h' too is in X.
(b) Let m be a moment of $\mathcal{B}$ and suppose that $\mathfrak{X}$ is a map that assigns to each agent a an element $\mathfrak{X}(a)$ of $Ch_a(m)$. Then there exists a history h that is contained in each $\mathfrak{X}(a)$.

The elements of $Ch_a(m)$ are said to be (*momentary*) *alternatives* of a at m, and $Ch_a(m)$ is said to be the *choice-set* of a at m. An agent a has *non-vacuous choice* at moment m if $Ch_a(m) \neq \{His_m\}$, i.e., if there are distinct $X, X' \in Ch_a(m)$.

By saying that each agent's choice-set forms a partition, we postulate that at each moment each agent is to choose exactly one of her/his alternatives. Condition (a) of 3.1 means that an agent cannot separate histories that are undivided at the moment of her/his choice. Finally, by condition (b), each agent can choose an alternative in her/his choice-set independently of which alternatives are chosen by all the other agents (at the same moment). In particular, at a given moment m no agent can prevent another agent from choosing any of her/his alternatives (at that moment).

DEFINITION 3.2 (T×W-based agent-frame). A T×W-*based agent-frame* is defined as an ordered triple $\mathcal{F} = \langle \mathcal{R}, Ag, f \rangle$ consisting of a T×W-frame $\mathcal{R} = \langle T, <, W, \sim \rangle$, a non-void set Ag of *agents*, and a map f that assigns to each agent $a \in Ag$, each time-point t, and each world w a subset $f_a(t, w)$ of the set

$$W^{t,w} := \left\{ w' \in W : w \sim_t w' \right\}$$

such that each of the following conditions is satisfied:

(a) For each $t \in T$ and each $w \in W$, $w \in f_a(t, w)$.
(b) From $f_a(t, w) \cap f_a(t, w') \neq \emptyset$ it follows that $f_a(t, w) = f_a(t, w')$.
(c) If $w \sim_{t'} w'$ for some $t' > t$, $f_a(t, w) = f_a(t, w')$.
(d) Let $(w_a)_{a \in Ag}$ be a family of worlds such that, for all $a, a' \in Ag$, $w_a \sim_t w_{a'}$. Then there is a world $w^* \in W$ with $w^* \in f_a(t, w_a)$ for each $a \in Ag$.

$f_a(t, w)$ is said to be *the alternative that is chosen by a at time-point t in world w*.

Heuristically and also technically (as can be seen from the following considerations), the conjunction of conditions (a) and (b) corresponds to the postulate that choice-sets are partitions, while (c) corresponds to condition (a) of Definition 3.1 and (d) to condition (b) of 3.1.

The concepts of agent-frame resting upon synchronized trees and T×W-frames, respectively, correspond to each other in the following sense: Let $\mathcal{F}$ be an agent-frame defined on a modally divergent and modally complete T×W-frame $\mathcal{R}$. Consider the synchronized tree $\mathcal{B}^{\mathcal{R}}$ that corresponds to $\mathcal{R}$

(as defined in Section 2). On the set of all histories of $\mathcal{B}^{\mathcal{R}}$ define an equivalence relation by

$$\begin{aligned} h_w \equiv_a^m h_{w'} \quad &:\Longleftrightarrow \quad w' \in f_a(t_m, w) \\ &\Longleftrightarrow \quad f_a(t_m, w) = f_a(t_m, w'), \end{aligned}$$

where $t_m := t$ if $m = [t, w]_\sim$, and let $Ch_a(m)$ be the set of all $\equiv_a^m$-equivalence classes in His_m. The partition $Ch_a(m)$ obviously satisfies condition (a) of 3.1. With regard to condition (b) we can argue as follows: Let $\mathfrak{X}$ be a map that assigns to each agent a an element $\mathfrak{X}(a) \in Ch_a(m)$. By the Axiom of Choice, choose for each $a \in Ag$ a world w_a with $h_{w_a} \in \mathfrak{X}(a)$. Hence, by condition (d) of 3.2, there is a world w^* with $w^* \in \bigcap_a f_a(t_m, w_a)$. Thus we get that, for each agent a, $h_{w^*} \equiv_a^m h_{w_a}$ and hence $h_{w^*} \in \mathfrak{X}(a)$.

Conversely, let $\mathcal{C}$ be an agent-frame defined on a synchronized tree $\mathcal{B}$ and let $\mathcal{R}^{\mathcal{B}}$ be the corresponding T×W-frame (cf. Section 2). Furthermore, let $f_a(i, h)$ be that element X of $Ch_a(m_{i,h})$ that contains h. Then it is only condition (d) of Definition 3.2 that is of interest: Let $(h_a)_{a \in Ag}$ be a family of worlds of $\mathcal{R}^{\mathcal{B}}$ with $h_a \sim_i h_{a'}$ for all $a, a' \in Ag$, and set $\mathfrak{X}(a) := f_a(i, h_a)$. Then there is an h^* such that, for each agent a, h^* is in $\mathfrak{X}(a)$ and hence in $f_a(i, h_a)$.

Finally, we introduce further concepts of agent-frame by weakening the independence condition. Thereto note that, for example, the independence condition of T×W-based agent-frames can be expressed equivalently by

(d*) Let A be an *arbitrary* non-void set of agents and let $\pi : A \to W$ be a map such that, for all $a, a' \in A$, $\pi(a) \sim_t \pi(a')$. Then there is a world w^π such that, for each $a \in A$, w^π is contained in $f_a(t, \pi(a))$.

DEFINITION 3.3 (Weak agent-frame). *Weak* T×W-*based agent-frames* are defined like T×W-based agent-frames in the sense of Definition 3.2, but with condition (d) replaced by a condition (d′) that postulates independence for *finite* sets of agents only (i.e., (d′) results from (d*) by replacing its beginning by "Let A be a *finite* non-void set of agents ..."). In an analogous manner the concept of *weak tree-based agent-frame* is introduced by weakening condition (b) of Definition 3.1 as follows:

(b′) Let A be a *finite* set of agents and let $\mathfrak{X}$ be a map that assigns to each agent $a \in A$ an element $\mathfrak{X}(a) \in Ch_a(m)$. Then there exists a history h with $h \in \bigcap_{a \in A} \mathfrak{X}(a)$.

In order to emphasize the difference between weak agent-frames and agent-frames in the normal sense, the latter ones may also be called *strong agent-frames*.

It should be clear that weak T×W-based and weak tree-based agent-frames correspond to each other via the transitions discussed above with respect to strong agent-frames. But there are also cases in which weak and strong agent-frames coincide. And one of these seems worth mentioning:

PROPOSITION 3.4. *Each weak agent-frame in which at each moment only finitely many agents have non-vacuous choice is strong.*

The premise of this proposition is more plausible than the stronger assumption that the set of agents itself is finite. Unfortunately, this premise (as is the stronger assumption) is not first-order axiomatizable. In Belnap et al. (2001), Chapter 17, Xu discusses conditions with regard to the number of alternatives that each agent has at most. As far as I see, there are no connections between such cardinality assumptions and the relationship between weak and strong agent-frames.

4. The Language $\mathcal{L}_{\mathsf{Q}}$ and Basic Semantical Concepts

In this section we introduce the formal language $\mathcal{L}_{\mathsf{Q}}$ upon which all our following considerations will rest and we discuss its basic semantical concepts, which are defined in terms of the frame-types presented in the previous section.

The alphabet of $\mathcal{L}_{\mathsf{Q}}$ contains countably many (individual) constants, denumerably many propositional constants, the symbols $=$, $\neg$, $\rightarrow$, G, H, N, $\mathsf{N}^{\neq}$, $\boxminus$, and Q, and parentheses. With respect to $\mathcal{L}_{\mathsf{Q}}$ the concept of *formula* is introduced in the usual manner: Propositional constants of $\mathcal{L}_{\mathsf{Q}}$ are formulae of $\mathcal{L}_{\mathsf{Q}}$. And, for all individual constants c and c' and all formulae φ and ψ of $\mathcal{L}_{\mathsf{Q}}$, $c = c'$, $\neg\varphi$, $(\varphi \rightarrow \psi)$, $\mathsf{G}\varphi$, $\mathsf{H}\varphi$, $\mathsf{N}\varphi$, $\mathsf{N}^{\neq}\varphi$, $\boxminus\varphi$, and $\mathsf{Q}c\,\varphi$ are formulae of $\mathcal{L}_{\mathsf{Q}}$.

G and H are the usual temporal operators: $\mathsf{G}\varphi$ means '*It will always be the case that φ is true*' and $\mathsf{H}\varphi$ stands for '*It has always been the case that φ is true*'. N is the operator of historical necessity ('*It is inevitable that* ...'). Sentences of the form $\boxminus\varphi$ may be read as '*In arbitrary circumstances it is now the case that φ is true*'. Finally, the operator $\mathsf{N}^{\neq}$ is an auxiliary operator that is introduced to improve the expressive power of the object language. It will occur in two contexts: in Section 6 this operator plays an essential role to establish the result that the canonical structure, which is defined there, satisfies condition (d′) of Definition 3.3 and in Section 7 it is used to axiomatize modal divergence of T×W-frames. The reading of $\mathsf{N}^{\neq}\varphi$ is '*In all merely possible worlds sharing the past and present with the actual world it is now the case that φ is true*'. As we will see below, the operator

N is definable by $\mathsf{N}^{\neq}$, but for historical reasons it will be considered as an undefined primitive.

Other temporal and modal operators are introduced as follows $\mathsf{F}\varphi := \neg\mathsf{G}\neg\varphi$, $\mathsf{P}\varphi := \neg\mathsf{H}\neg\varphi$, $\mathsf{M}\varphi := \neg\mathsf{N}\neg\varphi$, $\mathsf{M}^{\neq}\varphi := \neg\mathsf{N}^{\neq}\neg\varphi$, and $\diamondminus\varphi := \neg\boxminus\neg\varphi$. Connectives of propositional logic are defined in the usual manner as well as substitution operations (with respect to individual and propositional constants, respectively). For example, $\varphi(c'/c)$ is the formula that is obtained from formula φ by substituting the individual constant c' for each occurrence of c in φ.

The semantics of $\mathcal{L}_{\mathsf{Q}}$ is then defined by the following concepts of *agent-structure* and *validity*:

DEFINITION 4.1 (Agent-structures). (a) A (*weak*) T×W-*based agent-structure* is an ordered pair $\mathcal{A} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a (weak) T×W-based agent-frame (cf. definitions 3.2 and 3.3) and V is a map that assigns to each individual constant c an agent $|c|^{\mathcal{A}} := V(c) \in Ag$ and to each propositional constant p a subset $|p|^{\mathcal{A}} := V(p)$ of $T \times W$.

(b) A (*weak*) *tree-based agent-structure* is defined as an ordered pair $\mathcal{A} = \langle \mathcal{C}, V \rangle$ consisting of a (weak) tree-based agent-frame $\mathcal{C}$ (cf. definitions 3.1 and 3.3) and a map V that assigns to each individual constant c an agent $|c|^{\mathcal{A}} := V(c) \in Ag$ and to each propositional constant p a subset $|p|^{\mathcal{A}} := V(p)$ of $\{(m, h) \in Mom \times His : m \in h\}$.

Let now $\mathcal{A}$ be a (weak) T×W-based agent-structure. The concept of *validity in $\mathcal{A}$ at time-point t and world w* is defined recursively as follows:

$$
\begin{array}{lll}
\mathcal{A} \models_{t,w} p & \iff & (t, w) \in |p|^{\mathcal{A}}, \\
\mathcal{A} \models_{t,w} c = c' & \iff & |c|^{\mathcal{A}} = |c'|^{\mathcal{A}}, \\
\mathcal{A} \models_{t,w} \neg\varphi & \iff & \mathcal{A} \not\models_{t,w} \varphi, \\
\mathcal{A} \models_{t,w} \varphi \to \psi & \iff & \mathcal{A} \not\models_{t,w} \varphi \text{ or } \mathcal{A} \models_{t,w} \psi, \\
\mathcal{A} \models_{t,w} \mathsf{G}\varphi & \iff & \mathcal{A} \models_{t',w} \varphi, \text{ for all } t' > t, \\
\mathcal{A} \models_{t,w} \mathsf{H}\varphi & \iff & \mathcal{A} \models_{t',w} \varphi, \text{ for all } t' < t, \\
\mathcal{A} \models_{t,w} \mathsf{N}\varphi & \iff & \mathcal{A} \models_{t,w'} \varphi, \text{ for all } w' \sim_t w, \\
\mathcal{A} \models_{t,w} \mathsf{N}^{\neq}\varphi & \iff & \mathcal{A} \models_{t,w'} \varphi, \text{ for all } w' \sim_t w \text{ with } w \neq w', \\
\mathcal{A} \models_{t,w} \boxminus\varphi & \iff & \mathcal{A} \models_{t,w'} \varphi, \text{ for all } w' \in W, \\
\mathcal{A} \models_{t,w} \mathsf{Q}c\,\varphi & \iff & \mathcal{A} \models_{t,w'} \varphi, \text{ for all } w' \in f_{|c|^{\mathcal{A}}}(t, w).
\end{array}
$$

The corresponding validity concept with respect to (weak) tree-based agent-structures is defined analogously – note that $\models_{m,h}$ is only defined if m is element of h:

$$
\begin{array}{lll}
\mathcal{A} \models_{m,h} p & \iff & (m, h) \in |p|^{\mathcal{A}}, \\
\mathcal{A} \models_{m,h} c = c' & \iff & |c|^{\mathcal{A}} = |c'|^{\mathcal{A}}, \\
\mathcal{A} \models_{m,h} \neg\varphi & \iff & \mathcal{A} \not\models_{m,h} \varphi, \\
\mathcal{A} \models_{m,h} \varphi \to \psi & \iff & \mathcal{A} \not\models_{m,h} \varphi \text{ or } \mathcal{A} \models_{m,h} \psi,
\end{array}
$$

$$\begin{array}{lcl}
\mathcal{A} \models_{m,h} \mathsf{G}\varphi & \iff & \mathcal{A} \models_{m',h} \varphi,\ \text{for all } m' \in h \text{ with } m \prec m', \\
\mathcal{A} \models_{m,h} \mathsf{H}\varphi & \iff & \mathcal{A} \models_{m',h} \varphi,\ \text{for all } m' \prec m, \\
\mathcal{A} \models_{m,h} \mathsf{N}\varphi & \iff & \mathcal{A} \models_{m,h'} \varphi,\ \text{for all } h' \in His_m, \\
\mathcal{A} \models_{m,h} \mathsf{N}^{\neq}\varphi & \iff & \mathcal{A} \models_{m,h'} \varphi,\ \text{for all } h' \in His_m \text{ with } h \neq h', \\
\mathcal{A} \models_{m,h} \boxminus\varphi & \iff & \mathcal{A} \models_{m',h'} \varphi, \text{ for all } m' \in Mom \text{ such that there is an } i \in Inst \text{ with } m, m' \in i, \text{ and all } h' \in His_{m'}, \\
\mathcal{A} \models_{m,h} \mathsf{Q}c\,\varphi & \iff & \mathcal{A} \models_{m,h'} \varphi, \text{ for all } h' \in His_m \text{ such that there is an } X \in Ch_{|c|^{\mathcal{A}}}(m) \text{ with } h, h' \in X.
\end{array}$$

Further concepts of validity are introduced as usual. For example, a formula φ is said to be *valid* in a T×W-based agent-structure $\mathcal{A}$ if, for each $t \in T$ and each $w \in W$, it holds that $\mathcal{A} \models_{t,w} \varphi$. A set Φ of formulae is said to be *satisfiable* in a tree-based agent-structure $\mathcal{A}$ if there exist a moment m and a history $h \in His_m$ such that, for each $\varphi \in \Phi$, $\mathcal{A} \models_{m,h} \varphi$.

It is obvious that for both semantics appropriate coincidence and substitution theorems hold. We will refer to such theorems without proving them here. Summarizing the results of Section 3, we then obtain that the semantics defined in terms of tree-based agent-structures and the semantics given by modally complete and modally divergent T×W-based agent-structures yield the same logic:

THEOREM 4.2 (Correspondence Theorem). (a) *A formula φ of $\mathcal{L}_\mathsf{Q}$ is valid in each tree-based agent-structure if and only if φ is valid in each modally divergent and modally complete* T×W*-based agent-structure.*

(b) *A formula φ of $\mathcal{L}_\mathsf{Q}$ is valid in each weak* (T×W*-based, tree-based*) *agent-structure if and only if it is valid in each strong* (T×W*-based, tree-based*) *agent-structure.*

Proof. (a) Use the presented transitions from tree-based to T×W-based agent-frames, and vice versa. Furthermore, with regard to the definition of interpretation functions use the fact that there is a bijection between $T \times W$ and the set $\{(m, h) \in Mom \times His : m \in h\}$.

(b) Obviously, if φ is valid in each weak agent-structure, it is also valid in each strong agent-structure. Suppose now that there is a weak T×W-based agent-structure $\mathcal{A}$ with $\mathcal{A} \not\models_{t^*,w^*} \varphi$ for some $t^* \in T$ and some $w^* \in W$. There are only finitely many individual constants, say $c_1, \ldots, c_n$, that occur in φ. On the T×W-frame of $\mathcal{A}$ define now $Ag' := Ag$, $V' := V$, and

$$f'_a(t, w) := \begin{cases} f_a(t, w) & \text{if } a = |c_i|^{\mathcal{A}} \text{ for some } i \in \{1, \ldots, n\}, \\ W^{t,w} & \text{else.} \end{cases}$$

The agent-structure $\mathcal{A}'$ so defined is strong (cf. Proposition 3.4) and coincides with $\mathcal{A}$ at each pair (t, w) in the validation of all formulae in which only individual constants of $\{c_1, \ldots, c_n\}$ occur. The claim is proven, then, by applying an appropriate coincidence theorem. □

Note that the second claim of this theorem does not hold any longer if the language $\mathcal{L}_{\mathsf{Q}}$ is enriched by quantifiers. The formula $\forall x\, \mathsf{MQ}x\, \varphi \rightarrow \mathsf{M}\forall x\, \mathsf{Q}x\, \varphi$ (cf. Kutschera (1986)) is valid in strong agent-structures, but, in general, it is not valid in weak agent-structures. A short example may be helpful. Let us assume that there is a unary relation symbol F in the extended quantificational language. Now consider the agent-frame that is defined as follows: Let $W := \mathbb{N}$, $T := \{0\}$, $< := \emptyset$, $\sim_0 := W^2$, and $Ag := \mathbb{P}$ (where $\mathbb{N}$ is the set of natural numbers greater than 0 and $\mathbb{P}$ is the set of prime numbers). For each $p \in Ag$, we set

$$f_p(0, h) := \begin{cases} \{n \in \mathbb{N} : p \mid n\} & \text{if } p \mid h, \\ \mathbb{N} \setminus f_p(0, p) & \text{else.} \end{cases}$$

The frame such defined is a weak T×W-based agent-frame that is not strong (for cardinality reasons: if infinitely many agents have non-vacuous choice at some moment, the set of worlds passing through this moment is not countable). With regard to the proof of the weak independence condition we can argue so: let P be a finite non-void subset of $\mathbb{P}$ and, for each $p \in P$, choose a 'history' $n_p \in \mathbb{N}$. Consider then $P^* := \{p \in P : p \mid n_p\}$ and $n^* := \prod_{p \in P^*} p$ (i.e., $n^* = 1$ if P^* is void). It can easily be verified that, for each $p \in P$, n^* is element of $f_p(0, n_p)$ (note that the representation of n^* as product of prime numbers is unique). Let now $c_1, c_2, \ldots$ be an enumeration of all individual constants (we can assume that there are infinitely many) and let $\mathcal{A}$ be an agent-structure on this frame such that $V(c_p) = p$ and $V_{0,n}(F) = \{p \in \mathbb{P} : p \mid n\}$ for each $n \in \mathbb{N}$. Then it holds that $\mathcal{A} \models_{0,1} \forall x\, \mathsf{MQ}x\, Fx$, but not that $\mathcal{A} \models_{0,1} \mathsf{M}\forall x\, \mathsf{Q}x\, Fx$. To show the first relation, choose $p \in \mathbb{P}$ arbitrarily. We then obtain that $\mathcal{A} \models_{0,p} \mathsf{Q}c_p\, Fc_p$ and hence that $\mathcal{A} \models_{0,1} \mathsf{MQ}c_p\, Fc_p$. Thus, $\mathcal{A} \models_{0,1} \forall x\, \mathsf{MQ}x\, Fx$. But we also get that $\mathcal{A} \models_{0,1} \mathsf{N}\exists x\, \neg\mathsf{Q}x\, Fx$. For if $n \in \mathbb{N}$ is chosen arbitrarily, there exists a prime number $p > n$ ($\mathbb{P}$ has no upper bound). Hence, it holds that $f_p(0, n) = f_p(0, 1)$ and, consequently, that $\mathcal{A} \models_{0,n} \neg\mathsf{Q}c_p\, Fc_p$. From this we obtain that $\mathcal{A} \models_{0,n} \exists x\, \neg\mathsf{Q}x\, Fx$ and thus that $\mathcal{A} \models_{0,1} \mathsf{N}\exists x\, \neg\mathsf{Q}x\, Fx$.

5. THE CALCULUS $\mathfrak{K}_{\mathsf{Q}}$

We will now present the basic calculus $\mathfrak{K}_{\mathsf{Q}}$ for Q-logic.

AXIOMS OF $\mathfrak{K}_{\mathsf{Q}}$.

(A-0) Axioms of propositional logic.

(A-1) $c = c$.

(A-2) $c = c' \to (\varphi \to \varphi(c'/c))$.

(A-3) $\mathsf{G}(\varphi \to \psi) \to (\mathsf{G}\varphi \to \mathsf{G}\psi)$, $\mathsf{H}(\varphi \to \psi) \to (\mathsf{H}\varphi \to \mathsf{H}\psi)$, $\mathsf{N}(\varphi \to \psi) \to (\mathsf{N}\varphi \to \mathsf{N}\psi)$, $\mathsf{N}^{\neq}(\varphi \to \psi) \to (\mathsf{N}^{\neq}\varphi \to \mathsf{N}^{\neq}\psi)$, $\boxminus(\varphi \to \psi) \to (\boxminus\varphi \to \boxminus\psi)$, $\mathsf{Q}c\,(\varphi \to \psi) \to (\mathsf{Q}c\,\varphi \to \mathsf{Q}c\,\psi)$.

(A-4) $c = c' \to \mathsf{G}\,c = c'$, $c = c' \to \mathsf{H}\,c = c'$, $c = c' \to \boxminus\,c = c'$.

(A-5) $\mathsf{FH}\varphi \to \varphi$, $\mathsf{PG}\varphi \to \varphi$.

(A-6) $\mathsf{G}\varphi \to \mathsf{GG}\varphi$.

(A-7) $\mathsf{FP}\varphi \to \mathsf{P}\varphi \vee \varphi \vee \mathsf{F}\varphi$, $\mathsf{PF}\varphi \to \mathsf{P}\varphi \vee \varphi \vee \mathsf{F}\varphi$.

(A-8) $\mathsf{MN}\varphi \to \mathsf{N}\varphi$, $\neg\mathsf{Q}c\,\neg\mathsf{Q}c\,\varphi \to \mathsf{Q}c\,\varphi$, $\diamondminus\boxminus\,\varphi \to \boxminus\varphi$.

(A-9) $\mathsf{M}^{\neq}\mathsf{N}^{\neq}\varphi \to \varphi$.

(A-10) $\mathsf{N}\varphi \to \mathsf{Q}c\,\varphi$.

(A-11) $\mathsf{N}\varphi \to \mathsf{N}^{\neq}\varphi$.

(A-12) $\boxminus\varphi \to \mathsf{N}\varphi$.

(A-13) $\mathsf{Q}c\,\varphi \to \varphi$.

(A-14) $\mathsf{F}\boxminus\varphi \to \boxminus\mathsf{F}\varphi$, $\mathsf{P}\boxminus\varphi \to \boxminus\mathsf{P}\varphi$.

(A-15) $\mathsf{PQ}c\,\varphi \to \mathsf{NP}\varphi$.

(A-16) $\bigwedge_{0 \le i < j \le n} c_i \neq c_j \wedge \mathsf{Q}c_0\,(\neg\varphi_1 \vee \cdots \vee \neg\varphi_n) \to$
$\mathsf{N}(\mathsf{Q}c_1\,\varphi_1 \to \mathsf{N}(\mathsf{Q}c_2\,\varphi_2 \to \cdots \mathsf{N}(\mathsf{Q}c_{n-1}\,\varphi_{n-1} \to \mathsf{N}\neg\mathsf{Q}c_n\,\varphi_n)\cdots))$.

(A-17) $\psi \wedge \mathsf{N}^{\neq}\varphi \to \mathsf{N}(\neg\psi \to \varphi)$.

RULES OF $\mathfrak{K}_{\mathsf{Q}}$.

(R-MP) $\dfrac{\varphi, \varphi \to \psi}{\psi}$.

(R-G) $\dfrac{\varphi}{\mathsf{G}\varphi}$.

(R-H) $\dfrac{\varphi}{\mathsf{H}\varphi}$.

(R-$\boxminus$) $\dfrac{\varphi}{\boxminus\varphi}$.

(R-Irr) $\dfrac{\boxminus\vartheta_p \wedge \chi_q \to \varphi}{\varphi}$,

where p and q are distinct propositional constants that both do not occur in φ and where ϑ_p and χ_q are defined as $p \wedge \mathsf{G}\neg p \wedge \mathsf{H}\neg p$ and $q \wedge \mathsf{N}^{\neq}\neg q$, respectively.

The concepts of *proof w.r.t.* $\mathfrak{K}_{\mathsf{Q}}$, *provable from a set of formulae*, etc., are then introduced as usual.

By the calculus $\mathfrak{K}_{\mathsf{Q}}$, the operators N and $\boxminus$ are characterized as S5-necessities, while $\mathsf{N}^{\neq}$ is characterized as a mere B-necessity. And, provided $\mathsf{Q}c$ is understood as a unary operator, it is an S5-necessity, too. The calculus $\mathfrak{K}_{\mathsf{Q}}$ contains the temporal logic for linear time with respect to the operators G and H. The usual bridging principle for logical systems combining tense and modality is replaced by Axiom (A-15) and appears in the list of theorems below as (T-3). The necessity-rules for the operators N, $\mathsf{N}^{\neq}$, and $\mathsf{Q}c$ are derivable from (R-$\boxminus$) and axioms (A-10)–(A-12). The *irreflexivity rule* (R-Irr) is of special interest: In the completeness proof presented in Section 6 this rule provides us with *propositional names for time-points* and with *propositional names for worlds*. Note that the system of intensional operators of $\mathcal{L}_{\mathsf{Q}}$, i.e., the set $\{\mathsf{G}, \mathsf{H}, \boxminus, \mathsf{N}, \mathsf{N}^{\neq}\} \cup \{\mathsf{Q}c : c \text{ is an individual constant}\}$, is *dually complete*: G is the dual of H, and vice versa, and each of the operators $\boxminus$, N, $\mathsf{N}^{\neq}$, and $\mathsf{Q}c$ is the dual of itself. The proofs of some lemmata presented in Section 6 depend on the fact of dual completeness (cf. 6.2 and 6.5): this is the reason why the symmetry axiom for $\mathsf{N}^{\neq}$ is needed in the axiomatization. Thus, the rule (R-Irr) is the *reduction* of the following rule (R-Irr') (i.e., (R-Irr) and (R-Irr′) have the same proof theory on the basis of the other axioms and rules of $\mathfrak{K}_{\mathsf{Q}}$):

(R-Irr′) $\dfrac{\Box_1(\varphi_1 \to \cdots \Box_{n-1}(\varphi_{n-1} \to \Box_n(\boxminus\vartheta_p \wedge \chi_q \to \varphi_n))\cdots)}{\Box_1(\varphi_1 \to \cdots \Box_{n-1}(\varphi_{n-1} \to \Box_n\varphi_n)\cdots)}$,

where p and q are distinct propositional constants that both do not occur in any of the formulae φ_i, where the operators $\Box_i$ are arbitrary intensional operators, and where ϑ_p and χ_q are defined as $p \wedge \mathsf{G}\neg p \wedge \mathsf{H}\neg p$ and $q \wedge \mathsf{N}^{\neq}\neg q$, respectively.

THEOREMS OF $\mathfrak{K}_{\mathsf{Q}}$.

(T-1) $\mathsf{F}(\vartheta \wedge \varphi) \to \mathsf{G}(\vartheta \to \varphi)$, $\mathsf{P}(\vartheta \wedge \varphi) \to \mathsf{H}(\vartheta \to \varphi)$,
where ϑ *has the form* $\psi \wedge \mathsf{G}\neg\psi \wedge \mathsf{H}\neg\psi$.

(T-2) $\mathsf{HQ}c\,\varphi \to \mathsf{NH}\varphi$, $\mathsf{Q}c\,\mathsf{G}\varphi \to \mathsf{GN}\varphi$.

(T-3) $\mathsf{PN}\varphi \to \mathsf{NP}\varphi$, $\mathsf{HN}\varphi \to \mathsf{NH}\varphi$, $\mathsf{NG}\varphi \to \mathsf{GN}\varphi$.

(T-4) $\mathsf{G}\boxminus\varphi \leftrightarrow \boxminus\mathsf{G}\varphi$, $\mathsf{H}\boxminus\varphi \leftrightarrow \mathsf{H}\boxminus\varphi$.

(T-5) $\bigwedge_{1\le i<j\le n} c_i \neq c_j \wedge \mathsf{N}(\varphi_1 \wedge \cdots \wedge \varphi_{n-1} \to \varphi_n) \to$
$\mathsf{N}(\mathsf{Q}c_1\,\varphi_1 \to \cdots \mathsf{N}(\mathsf{Q}c_{n-1}\,\varphi_{n-1} \to \mathsf{N}\neg\mathsf{Q}c_n\,\neg\varphi_n)\cdots)$.

(T-6) $\bigwedge_{1\le i<j\le n} c_i \neq c_j \wedge \mathsf{MQ}c_1\,\varphi_1 \wedge \cdots \wedge \mathsf{MQ}c_n\,\varphi_n \to$
$\mathsf{M}(\mathsf{Q}c_1\,\varphi_1 \wedge \cdots \wedge \mathsf{Q}c_n\,\varphi_n)$.

(T-7) $c \neq c' \to (\mathsf{N}\varphi \leftrightarrow \mathsf{Q}c\,\mathsf{Q}c'\,\varphi)$.

(T-8) $\chi \wedge \varphi \to \mathsf{N}(\chi \to \varphi)$, *where* χ *has the form* $\psi \wedge \mathsf{N}^{\neq}\neg\psi$.

(T-9) $\mathsf{N}\varphi \leftrightarrow \varphi \wedge \mathsf{N}^{\neq}\varphi$.

Proof. Ad (T-5): We sketch the proof for $n = 3$: $\mathsf{MN}(\varphi_1 \wedge \varphi_2 \to \varphi_3)$ implies $\mathsf{N}(\varphi_1 \wedge \varphi_2 \to \varphi_3)$ and the latter formula implies $\mathsf{Q}c_1\,\varphi_1 \to \mathsf{Q}c_1\,(\varphi_2 \to \varphi_3)$. By Axiom (A-16), we then have $\mathsf{Q}c_1\,(\varphi_2 \to \varphi_3) \to \mathsf{N}(\mathsf{Q}c_2\,\varphi_2 \to \mathsf{N}\neg\mathsf{Q}c_3\,\neg\varphi_3)$ provided that $\bigwedge_{1\le i<j\le 3} c_i \neq c_j$. Thus, we obtain that the formula

$$\begin{aligned}&\mathsf{MN}(\varphi_1 \wedge \varphi_2 \to \varphi_3) \wedge \textstyle\bigwedge_{1\le i<j\le 3} c_i \neq c_j \to \\ &\quad \big(\mathsf{Q}c_1\,\varphi_1 \to \mathsf{N}(\mathsf{Q}c_2\,\varphi_2 \to \mathsf{N}\neg\mathsf{Q}c_3\,\neg\varphi_3)\big)\end{aligned}$$

is provable and hence, by (A-4) and (A-8), so is

$$\begin{aligned}&\textstyle\bigwedge_{1\le i<j\le 3} c_i \neq c_j \wedge \mathsf{N}(\varphi_1 \wedge \varphi_2 \to \varphi_3) \to \\ &\quad \mathsf{N}\big(\mathsf{Q}c_1\,\varphi_1 \to \mathsf{N}(\mathsf{Q}c_2\,\varphi_2 \to \mathsf{N}\neg\mathsf{Q}c_3\,\neg\varphi_3)\big).\end{aligned}$$

Ad (T-6): We sketch the proof for $n = 2$: By (T-5), we obtain that

$$\begin{aligned}&c_1 \neq c_2 \wedge \mathsf{N}(\mathsf{Q}c_1\,\varphi_1 \to \neg\mathsf{Q}c_2\,\varphi_2) \to \\ &\quad \mathsf{N}(\mathsf{Q}c_1\,\mathsf{Q}c_1\,\varphi_1 \to \mathsf{N}\neg\mathsf{Q}c_2\,\mathsf{Q}c_2\,\varphi_2)\end{aligned}$$

is provable. From the consequent of this implication it follows $\mathsf{Q}c_1\,\varphi_1 \to \mathsf{N}\neg\mathsf{Q}c_2\,\varphi_2$. Hence, the formula

$$c_1 \neq c_2 \wedge \mathsf{Q}c_1\,\varphi_1 \to \big(\mathsf{N}(\mathsf{Q}c_1\,\varphi_1 \to \neg\mathsf{Q}c_2\,\varphi_2) \to \mathsf{N}\neg\mathsf{Q}c_2\,\varphi_2\big)$$

is provable and, by contraposition, so is

$$c_1 \neq c_2 \wedge \mathsf{Q}c_1\,\varphi_1 \to \big(\mathsf{MQ}c_2\,\varphi_2 \to \mathsf{M}(\mathsf{Q}c_1\,\varphi_1 \wedge \mathsf{Q}c_2\,\varphi_2)\big).$$

Then, by some S5-modal logic w. r. t. N, the formula

$$c_1 \neq c_2 \wedge \mathsf{Q}c_1\,\varphi_1 \to \mathsf{N}\big(\mathsf{MQ}c_2\,\varphi_2 \to \mathsf{M}(\mathsf{Q}c_1\,\varphi_1 \wedge \mathsf{Q}c_2\,\varphi_2)\big),$$

is provable and thus so is

$$c_1 \neq c_2 \wedge \mathsf{MQ}c_1\,\varphi_1 \to \big(\mathsf{MQ}c_2\,\varphi_2 \to \mathsf{M}(\mathsf{Q}c_1\,\varphi_1 \wedge \mathsf{Q}c_2\,\varphi_2)\big).$$

Ad (T-7): By (A-10), the implication $\mathsf{N}\varphi \to \mathsf{Q}c\,\mathsf{Q}c'\,\varphi$ holds almost trivially. Conversely, by Axiom (A-16), we obtain that $c \neq c' \wedge \mathsf{Q}c\,\mathsf{Q}c'\,\varphi \to \mathsf{N}\neg\mathsf{Q}c'\,\neg\mathsf{Q}c'\,\varphi$ is provable and hence, by (A-8) and (A-13), so is $c \neq c' \wedge \mathsf{Q}c\,\mathsf{Q}c'\,\varphi \to \mathsf{N}\varphi$.

Ad (T-8): By Axiom (A-17), we obtain that $\psi \wedge \mathsf{N}^{\neq}\neg\psi \wedge \varphi \to \mathsf{N}(\neg(\psi \wedge \varphi) \to \neg\psi)$ is provable. But from the consequent of this formula it follows first $\mathsf{N}(\psi \to \varphi)$ and then $\mathsf{N}(\psi \wedge \mathsf{N}^{\neq}\neg\psi \to \varphi)$. □

LEMMA 5.1. *Let $\mathcal{A}$ be a weak* T×W*-based agent-structure, let ϑ be a formula of the form $\varphi \wedge \mathsf{H}\neg\varphi \wedge \mathsf{G}\neg\varphi$, and let χ be a formula of the form $\psi \wedge \mathsf{N}^{\neq}\neg\psi$. Then, for all time-points t and t' and all worlds w and w' of $\mathcal{A}$ with $w \sim_t w'$, the following implications hold:*

$$\mathcal{A} \models_{t,w} \vartheta \implies (\mathcal{A} \models_{t',w} \vartheta \iff t = t'),$$
$$\mathcal{A} \models_{t,w} \chi \implies (\mathcal{A} \models_{t,w'} \chi \iff w = w').$$

THEOREM 5.2 (Soundness of $\mathfrak{K}_{\mathsf{Q}}$). *Each formula provable in the calculus $\mathfrak{K}_{\mathsf{Q}}$ is valid in each* (*weak/strong*) T×W*-based agent-structure and hence it is also valid in each* (*weak/strong*) *tree-based agent-structure.*

Proof. Let $\mathcal{A} = \langle \mathcal{F}, V \rangle$ be a weak T×W-based agent-structure and define $[\varphi]_t^{\mathcal{A}} := \{w : \mathcal{A} \models_{t,w} \varphi\}$.

Ad (A-9): Suppose that $\mathcal{A} \models_{t,w} \mathsf{M}^{\neq}\mathsf{N}^{\neq}\varphi$. Then there is a world $w' \sim_t w$ distinct from w such that, for each world $w'' \sim_t w', \neq w'$, it holds that $\mathcal{A} \models_{t,w''} \varphi$. From this we obtain immediately that $\mathcal{A} \models_{t,w} \varphi$.

Ad (A-15): Suppose that $\mathcal{A} \models_{t,w} \mathsf{PQ}c\,\varphi$ and that $w' \sim_t w$. Then there exists a time-point $t' < t$ such that, for each $w'' \in f_{|c|^{\mathcal{A}}}(t', w)$, $\mathcal{A} \models_{t',w''} \varphi$. By condition (c) of 3.2, we obtain that $f_{|c|^{\mathcal{A}}}(t', w)$ and $f_{|c|^{\mathcal{A}}}(t', w')$ are identical. Hence, w' is in $f_{|c|^{\mathcal{A}}}(t', w)$, consequently $\mathcal{A} \models_{t',w'} \varphi$, and thus $\mathcal{A} \models_{t,w'} \mathsf{P}\varphi$.

Ad (A-16): Suppose that $\mathcal{A} \models_{t,w} \mathsf{Q}c_0\,(\neg\varphi_1 \vee \cdots \vee \neg\varphi_n)$ and let w_0 be w. Consider then $w_0 \sim_t w_1 \sim_t \ldots \sim_t w_n$ such that, for each $i \in \{1, \ldots, n-1\}$, $\mathcal{A} \models_{t,w_i} \mathsf{Q}c_i\,\varphi_i$. By condition (d′) of 3.3, there is a world w^* of $\mathcal{A}$ with $w^* \in f_{|c_0|^{\mathcal{A}}}(t, w_0) \cap \cdots \cap f_{|c_n|^{\mathcal{A}}}(t, w_n)$. From $f_{|c_0|^{\mathcal{A}}}(t, w_0) \subseteq [\neg\varphi_1]_t^{\mathcal{A}} \cup \cdots \cup [\neg\varphi_n]_t^{\mathcal{A}}$ it follows that there is a $j \in \{1, \ldots, n\}$ such that $\mathcal{A} \models_{t,w^*} \neg\varphi_j$. But, for each $j \in \{1, \ldots, n-1\}$, it holds that w^* is in $f_{|c_j|^{\mathcal{A}}}(t, w_j) \subseteq [\varphi_j]_t^{\mathcal{A}}$, and hence w^* must be in $[\neg\varphi_n]_t^{\mathcal{A}}$. From $w^* \in f_{|c_n|^{\mathcal{A}}}(t, w_n)$ we obtain that $f_{|c_n|^{\mathcal{A}}}(t, w_n) \not\subseteq [\varphi_n]_t^{\mathcal{A}}$ and thus that $\mathcal{A} \models_{t,w_n} \neg\mathsf{Q}c_n\,\varphi_n$.

Ad (R-Irr): Assume that $\mathcal{A} \not\models_{t^*,w^*} \varphi$. Let p and q be distinct propositional constants that both do not occur in φ. Define an agent-structure $\mathcal{A}' = \langle \mathcal{F}, V' \rangle$ as follows: $V'(p) := \{(t^*, w) : w \in W\}$, $V'(q) := \{(t, w^*) : t \in T\}$, and $V'(r) := V(r)$ for each propositional constant r

distinct from p and q. By an appropriate coincidence theorem, we obtain that $\mathcal{A}' \not\models_{t^*,w^*} \varphi$. It is easy to verify, then, that $\mathcal{A}' \models_{t^*,w^*} \boxminus\vartheta_p$ and that $\mathcal{A}' \models_{t^*,w^*} \chi_q$. Hence we get that $\mathcal{A}' \not\models_{t^*,w^*} \boxminus\vartheta_p \wedge \chi_q \to \varphi$. □

6. Completeness of $\mathfrak{K}_{\mathsf{Q}}$

In this section we will establish that the calculus $\mathfrak{K}_{\mathsf{Q}}$ is strongly complete with respect to weak T×W-based agent-structures, i.e., each set of formulae consistent w.r.t. $\mathfrak{K}_{\mathsf{Q}}$ is satisfiable in a weak T×W-based agent-structure. This result is obtained by an appropriate mixture of techniques that are standard in the literature. Hence, in what follows it is presupposed that the reader is familiar with usual concepts such as *consistency* and *maximal consistency*. To keep our considerations widely general, let $\mathcal{L}'_{\mathsf{Q}}$ be an arbitrary extension of $\mathcal{L}_{\mathsf{Q}}$ (i.e., $\mathcal{L}'_{\mathsf{Q}}$ results from $\mathcal{L}_{\mathsf{Q}}$ by adding new propositional constants and/or individual constants). Let $P(\mathcal{L}'_{\mathsf{Q}})$ denote the set of propositional constants of $\mathcal{L}'_{\mathsf{Q}}$ and let $C(\mathcal{L}'_{\mathsf{Q}})$ be the set of individual constants of $\mathcal{L}'_{\mathsf{Q}}$. We use the notation $\varphi \in \mathcal{L}'_{\mathsf{Q}}$ as an abbreviation of 'φ is a formula of $\mathcal{L}'_{\mathsf{Q}}$'. Note that each consistent theory of $\mathcal{L}_{\mathsf{Q}}$ (i.e., each consistent set of formulae of $\mathcal{L}_{\mathsf{Q}}$) is also consistent if considered as an $\mathcal{L}'_{\mathsf{Q}}$-theory. As *intensional operators* of $\mathcal{L}'_{\mathsf{Q}}$ we refer to operators in $\{\mathsf{G}, \mathsf{H}, \mathsf{N}, \mathsf{N}^{\neq}, \boxminus\} \cup \{\mathsf{Q}c : c \in C(\mathcal{L}'_{\mathsf{Q}})\}$. For each of these operators, $\Box$, let $\Diamond\varphi$ be the abbreviation of $\neg\Box\neg\varphi$.

DEFINITION 6.1. A maximally consistent theory Σ of $\mathcal{L}'_{\mathsf{Q}}$ is said to be an *irreflexive theory* if each of the following conditions is satisfied:

(a) There are distinct propositional constants p and q with $\boxminus\vartheta_p \wedge \chi_q \in \Sigma$.
(b) Let $\Box_1, \ldots, \Box_n$ be intensional operators and let $\varphi_1, \ldots, \varphi_n$ be formulae of $\mathcal{L}'_{\mathsf{Q}}$ such that Σ contains the formula

$$\Box_1\big(\varphi_1 \to \Box_2\big(\varphi_2 \to \cdots \Box_{n-1}\big(\varphi_{n-1} \to \Box_n(\boxminus\vartheta_p \wedge \chi_q \to \varphi_n)\big) \cdots\big)\big)$$

for all distinct propositional constants p and q such that neither of which occurs in any of the formulae $\varphi_1, \ldots, \varphi_n$. Then Σ also contains the formula

$$\Box_1\big(\varphi_1 \to \Box_2\big(\varphi_2 \to \cdots \Box_{n-1}(\varphi_{n-1} \to \Box_n\varphi_n) \cdots\big)\big).$$

LEMMA 6.2. *Each consistent theory of $\mathcal{L}'_{\mathsf{Q}}$ in which infinitely many propositional constants of $\mathcal{L}'_{\mathsf{Q}}$ do not occur can be extended to an irreflexive theory of $\mathcal{L}'_{\mathsf{Q}}$.*

Proof. Cf. Lemma 6.2.4 in Gabbay et al. (1994), p. 184. □

Let now Ω be a non-void set of irreflexive theories. For each intensional operator $\Box$, we set in the usual manner

$$\Sigma\ R_{\Box}\ \Sigma' :\Longleftrightarrow \left\{\varphi \in \mathcal{L}'_{\mathsf{Q}} : \Box\varphi \in \Sigma\right\} \subseteq \Sigma',$$

where Σ and Σ' are arbitrary irreflexive theories. Obviously, each of the relations R_{N}, $R_{\boxminus}$, and $R_c := R_{\mathsf{Q}c}$ (where c is an individual constant of $\mathcal{L}'_{\mathsf{Q}}$) is an equivalence relation on Ω. Furthermore, it holds that, for each individual constant c of $\mathcal{L}'_{\mathsf{Q}}$,

$$R_c \subseteq R_{\mathsf{N}} \subseteq R_{\boxminus}.$$

The relation R_{G} is transitive, linear-to-the-left, and linear-to-the-right on Ω (and even irreflexive: cf. Lemma 6.3(a) below). R_{H} is the inverse relation of R_{G}. Thus, we obtain an equivalence relation on Ω by

$$\Sigma \simeq \Sigma' :\Longleftrightarrow \Sigma\ R_{\mathsf{G}}\ \Sigma' \text{ or } \Sigma = \Sigma' \text{ or } \Sigma\ R_{\mathsf{H}}\ \Sigma'.$$

Finally, by (A-11) and (A-17), we get that

$$\Sigma\ R_{\mathsf{N}\neq}\ \Sigma' \Longleftrightarrow \Sigma\ R_{\mathsf{N}}\ \Sigma' \text{ and } \Sigma \neq \Sigma'.$$

LEMMA 6.3. *Let Ω be a non-void set of irreflexive theories of $\mathcal{L}'_{\mathsf{Q}}$.*

(a) *Provided $\Sigma, \Sigma' \in \Omega$ are equivalent w.r.t. $\simeq$ and provided there is formula of the form $\varphi \wedge \mathsf{G}\neg\varphi \wedge \mathsf{H}\neg\varphi$ that is contained in both theories, Σ and Σ' are identical.*
(b) *Provided $\Sigma, \Sigma' \in \Omega$ are R_{N}-equivalent and provided there is a formula $\varphi \wedge \mathsf{N}^{\neq}\neg\varphi$ in both theories, Σ and Σ' are identical.*
(c) *The relation R_{G} is a linear order on each equivalence class w.r.t. $\simeq$.*

In what follows we will investigate some properties of specific sets of irreflexive theories.

DEFINITION 6.4. Let Ω be a non-void set of irreflexive theories of $\mathcal{L}'_{\mathsf{Q}}$ and let C be a set of individual constants of $\mathcal{L}'_{\mathsf{Q}}$.

(a) Ω is said to be *closed* if for each $\Sigma \in \Omega$, each intensional operator $\Box$ of $\mathcal{L}'_{\mathsf{Q}}$, and each formula $\Diamond\varphi \in \Sigma$ there is an irreflexive theory $\Sigma' \in \Omega$ with $\Sigma\ R_{\Box}\ \Sigma'$ and $\varphi \in \Sigma'$.
(b) Ω is said to be *connected* if for all $\Sigma, \Sigma' \in \Omega$ there exist $\Gamma, \Gamma' \in \Omega$ with $\Sigma \simeq \Gamma$, $\Sigma' \simeq \Gamma'$, and $\Gamma\ R_{\boxminus}\ \Gamma'$.
(c) Ω is said to be *maximally connected* if Ω is connected and if there is no connected set Ω' that contains Ω as proper subset.
(d) Ω is said to be *C-independent* if for each map $\pi : C \to \Omega$ such that, for all $c, c' \in C$, $\pi(c)\ R_{\mathsf{N}}\ \pi(c')$ there is a $\Sigma^{\pi} \in \Omega$ with $\pi(c)\ R_c\ \Sigma^{\pi}$ for each $c \in C$.

Let now Ω be a connected set of irreflexive theories of $\mathcal{L}'_{\mathsf{Q}}$. Then the following equivalence holds for all individual constants $c, c' \in C(\mathcal{L}'_{\mathsf{Q}})$ and all $\Sigma, \Sigma' \in \Omega$ (cf. Axiom (A-4)):

$$c = c' \in \Sigma \iff c = c' \in \Sigma'.$$

Hence, we get an equivalence relation on $C(\mathcal{L}'_{\mathsf{Q}})$ by

$$c \cong^{\Omega} c' :\iff c = c' \in \Sigma, \text{ for some/all } \Sigma \in \Omega.$$

A set C of individual constants is said to be Ω-*disjoint* if, for all $c, c' \in C$, the $\cong^{\Omega}$-equivalence classes of c and c' are disjoint. We say that Ω is (*weakly*) $\mathcal{L}'_{\mathsf{Q}}$-*independent* if Ω is C-independent for each (finite) Ω-disjoint set C of individual constants of $\mathcal{L}'_{\mathsf{Q}}$.

LEMMA 6.5. *The set of all irreflexive theories of $\mathcal{L}'_{\mathsf{Q}}$ is closed.*

Proof. Cf., for example, Lemma 6.2.5 in Gabbay et al. (1994), pp. 184–185. □

LEMMA 6.6. *Let Ω be a closed set of irreflexive theories of $\mathcal{L}'_{\mathsf{Q}}$.*

(a) *Let $\Sigma, \Sigma' \in \Omega$ with $\Sigma\ R_{\boxminus}\ \Sigma'$. Then for each $\Gamma \in \Omega$ with $\Gamma\ R_{\mathsf{G}}\ \Sigma$ there exists exactly one irreflexive theory $\Gamma' \in \Omega$ with $\Gamma'\ R_{\mathsf{G}}\ \Sigma'$ and $\Gamma\ R_{\boxminus}\ \Gamma'$.*
(b) *Let $\Gamma, \Gamma' \in \Omega$ with $\Gamma\ R_{\boxminus}\ \Gamma'$. Then for each $\Sigma \in \Omega$ with $\Gamma\ R_{\mathsf{G}}\ \Sigma$ there exists exactly one irreflexive theory $\Sigma' \in \Omega$ with $\Gamma'\ R_{\mathsf{G}}\ \Sigma'$ and $\Sigma\ R_{\boxminus}\ \Sigma'$.*

Proof. Cf., for example, Proposition 5.4 in Wölfl (1999a). □

LEMMA 6.7. *Let Ω be a closed set of irreflexive theories of $\mathcal{L}'_{\mathsf{Q}}$.*

(a) *Each maximally connected subset of Ω is closed.*
(b) *The set of all maximally connected subsets of Ω is a partition of Ω.*

Proof. (a) Let Ω' be a maximally connected subset of Ω. With respect to the operator N, for example, we can then argue as follows: Let $\Gamma \in \Omega'$ and $\mathsf{M}\gamma \in \Gamma$. Since Ω is closed, there exists an irreflexive theory $\Gamma^* \in \Omega$ with $\Gamma\ R_{\mathsf{N}}\ \Gamma^*$ and $\gamma \in \Gamma^*$. Then, $\Gamma\ R_{\boxminus}\ \Gamma^*$ and hence, since Ω' is maximally connected, $\Gamma^* \in \Omega'$.

(b) First, each $\Sigma \in \Omega$ is contained in a maximally connected subset of Ω. By Zorn's Lemma, this follows immediately from the fact that the set of all those connected subsets of Ω to which Σ belongs is inductively ordered by the subset relation. Second, suppose that there are maximally connected subsets Ω', Ω'' of Ω with non-void intersection. By Lemma 6.6, it can be verified that $\Omega' \cup \Omega''$ is connected. Since both Ω' and Ω'' are maximally connected, it then follows that Ω' and Ω'' are identical. □

In what follows, let Ω be a fixed maximally connected and (weakly) $\mathcal{L}'_{\mathsf{Q}}$-independent set of irreflexive theories (of $\mathcal{L}'_{\mathsf{Q}}$). Then, by Lemma 6.5 and Lemma 6.7, Ω is closed, too.

Let $\Xi, \Xi', \ldots$ denote equivalence classes w.r.t. $\boxminus$ that are included in Ω and let T^{Ω} be the set of all such equivalence classes. Analogously, let $\Psi, \Psi', \ldots$ designate $\simeq$-equivalence classes that are subsets of Ω and let W^{Ω} be the set of all these classes.

PROPOSITION 6.8. *For each* $\Xi \in T^{\Omega}$ *and each* $\Psi \in W^{\Omega}$ *there exists exactly one irreflexive theory* $\Sigma_{\Xi,\Psi} \in \Omega$ *with* $\Sigma_{\Xi,\Psi} \in \Xi \cap \Psi$. *And, for all* $\Xi, \Xi' \in T^{\Omega}$ *and all* $\Psi, \Psi' \in W^{\Omega}$*, it holds:*

$$\Sigma_{\Xi,\Psi}\ R_{\mathsf{G}}\ \Sigma_{\Xi',\Psi} \Longrightarrow \Sigma_{\Xi,\Psi'}\ R_{\mathsf{G}}\ \Sigma_{\Xi',\Psi'}.$$

Proof. This follows from the definition of connectivity, Lemma 6.3, and Lemma 6.6. □

Using the notations of Proposition 6.8, we can then define *the canonical* T×W*-frame with respect to* Ω, $\mathcal{R}^{\Omega} := \langle T^{\Omega}, <^{\Omega}, W^{\Omega}, \sim^{\Omega}\rangle$, by setting[4]

$$\begin{aligned} \Xi <^{\Omega} \Xi' &:\Longleftrightarrow \Sigma_{\Xi,\Psi}\ R_{\mathsf{G}}\ \Sigma_{\Xi',\Psi}, \text{ for some/all } \Psi \in T^{\Omega}, \\ \Psi \sim^{\Omega}_{\Xi} \Psi' &:\Longleftrightarrow \Sigma_{\Xi,\Psi}\ R_{\mathsf{N}}\ \Sigma_{\Xi,\Psi'}. \end{aligned}$$

We will now see how an agent-frame can be constructed on $\mathcal{R}^{\Omega}$. To begin, we define Ag^{Ω} as the set of all equivalence classes w.r.t. $\cong^{\Omega}$. Obviously, this set is non-void. Then we continue by defining

$$f^{\Omega}_{\overline{c}}(\Xi, \Psi) := \left\{\Psi' \in W^{\Omega} : \Sigma_{\Xi,\Psi}\ R_c\ \Sigma_{\Xi,\Psi'}\right\},$$

where $c \in C(\mathcal{L}'_{\mathsf{Q}})$ and where $\overline{c}$ denotes the equivalence class of c w.r.t. $\cong^{\Omega}$. This is well-defined (cf. (A-2)). Furthermore, by (A-10), we obtain that

$$f^{\Omega}_{\overline{c}}(\Xi, \Psi) \subseteq \left\{\Psi' \in W^{\Omega} : \Psi \sim^{\Omega}_{\Xi} \Psi'\right\}.$$

THEOREM 6.9. *Let* Ω *be a maximally connected and* (*weakly*) $\mathcal{L}'_{\mathsf{Q}}$*-independent set of irreflexive theories. Then the triple* $\mathcal{F}^{\Omega} := \langle \mathcal{R}^{\Omega}, Ag^{\Omega}, f^{\Omega}\rangle$ *is a* (*weak*) T×W*-based agent-frame.*

Proof. We have to show that all the conditions of Definition 3.2 (of Definition 3.3, respectively) are satisfied.

(a) By Axiom (A-13), $\Psi \in f^{\Omega}_{\overline{c}}(\Xi, \Psi)$ for each $\Psi \in W^{\Omega}$.

(b) Let $\Psi_1, \Psi_2 \in W^\Omega$ and $\Psi' \in f_{\overline{c}}^\Omega(\Xi, \Psi_1) \cap f_{\overline{c}}^\Omega(\Xi, \Psi_2)$. Hence,

$$\Sigma_{\Xi,\Psi_1}\ R_c\ \Sigma_{\Xi,\Psi'} \quad \text{and} \quad \Sigma_{\Xi,\Psi_2}\ R_c\ \Sigma_{\Xi,\Psi'}.$$

For each $\Psi \in f_{\overline{c}}^\Omega(\Xi, \Psi_1)$ it holds that $\Sigma_{\Xi,\Psi_1}\ R_c\ \Sigma_{\Xi,\Psi}$, hence that

$$\Sigma_{\Xi,\Psi_2}\ R_c\ \Sigma_{\Xi,\Psi'}\ R_c\ \Sigma_{\Xi,\Psi_1}\ R_c\ \Sigma_{\Xi,\Psi},$$

and consequently that $\Psi \in f_{\overline{c}}^\Omega(\Xi, \Psi_2)$. Thus, $f_{\overline{c}}^\Omega(\Xi, \Psi_1)$ is a subset of $f_{\overline{c}}^\Omega(\Xi, \Psi_2)$. The other inclusion follows analogously.

(c) Let $\Psi, \Psi' \in W^\Omega$ with $\Psi \sim_{\Xi'}^\Omega \Psi'$ for some $\Xi' >^\Omega \Xi$. We have to show that $f_{\overline{c}}^\Omega(\Xi, \Psi)$ and $f_{\overline{c}}^\Omega(\Xi, \Psi')$ are identical. For this it is sufficient to prove that the relation $\Sigma_{\Xi,\Psi}\ R_c\ \Sigma_{\Xi,\Psi'}$ holds (if this is established, it follows $\Psi' \in f_{\overline{c}}^\Omega(\Xi, \Psi) \cap f_{\overline{c}}^\Omega(\Xi, \Psi')$ and hence, by (b), the assertion). Thus, suppose that $\mathsf{Q}c\,\varphi$ is in $\Sigma_{\Xi,\Psi}$. By irreflexivity of $\Sigma_{\Xi,\Psi}$, there is a propositional constant p such that $\Sigma_{\Xi,\Psi}$ contains the formula $\boxminus\vartheta_p$ and hence, by (A-12) and (A-10), $\mathsf{Q}c\,(\vartheta_p \wedge \varphi)$, too. Consequently, $\mathsf{PQ}c\,(\vartheta_p \wedge \varphi)$ and hence, by (A-15), $\mathsf{NP}(\vartheta_p \wedge \varphi)$ are in $\Sigma_{\Xi',\Psi}$. Then $\mathsf{P}(\vartheta_p \wedge \varphi)$ is contained in $\Sigma_{\Xi',\Psi'}$ and hence, by (T-1), so is $\mathsf{H}(\vartheta_p \to \varphi)$. Since $\Sigma_{\Xi,\Psi'}\ R_{\mathsf{G}}\ \Sigma_{\Xi',\Psi'}$ and $\vartheta_p \in \Sigma_{\Xi,\Psi'}$, we obtain that φ is in $\Sigma_{\Xi,\Psi'}$ – as has been claimed.

(d/d′) Let A be a (finite) non-void set of 'agents' of $\mathcal{F}^\Omega$, i.e., a (finite) set of equivalence classes w.r.t. $\cong^\Omega$. Consider $\Xi \in T^\Omega$ and let $\pi : A \to W^\Omega$ be a map such that, for all $a, a' \in A$, $\pi(a) \sim_\Xi^\Omega \pi(a')$. Choose in each equivalence class $a \in A$ an individual constant c_a. Then we have $\overline{c_a} = a$, and the set $C_A := \{c_a : a \in A\}$ of individual constants is Ω-disjoint. Define $\pi'(c) := \Sigma_{\Xi,\pi(\overline{c})}$. Since Ω is (weakly) $\mathcal{L}'_{\mathsf{Q}}$-independent, there is a $\Sigma^{\pi'} \in \Omega$ with $\Sigma^\pi\ R_c\ \pi'(c)$. Then let Ψ^π be that element of W^Ω that contains $\Sigma^{\pi'}$. By definitions, it then follows that, for each $c \in C_A$, $\Psi^\pi \in f_{\overline{c}}^\Omega(\Xi, \pi(\overline{c}))$ and hence that, for each $a \in A$, $\Psi^\pi \in f_a^\Omega(\Xi, \pi(a))$. □

Thus, the critical point of the completeness proof is to show that for each consistent set Σ of $\mathcal{L}_{\mathsf{Q}}$ there exist an extension $\mathcal{L}'_{\mathsf{Q}}$ of $\mathcal{L}_{\mathsf{Q}}$ and an irreflexive $\mathcal{L}'_{\mathsf{Q}}$-theory Σ' such that the maximally connected set of $\mathcal{L}'_{\mathsf{Q}}$-theories to which Σ' belongs is (weakly) $\mathcal{L}'_{\mathsf{Q}}$-independent. In what follows, let $\mathcal{L}^*_{\mathsf{Q}}$ be an extension of $\mathcal{L}_{\mathsf{Q}}$ that is obtained from $\mathcal{L}_{\mathsf{Q}}$ by adding denumerably many new propositional constants. Then each consistent theory of $\mathcal{L}_{\mathsf{Q}}$ is contained in an irreflexive theory of $\mathcal{L}^*_{\mathsf{Q}}$. Furthermore, we have

LEMMA 6.10. *Each maximally connected set Ω of $\mathcal{L}^*_{\mathsf{Q}}$-theories is weakly $\mathcal{L}^*_{\mathsf{Q}}$-independent.*

Before proving this lemma, we insert a preliminary remark:

LEMMA 6.11. *Let C be an Ω-disjoint set of individual constants of $\mathcal{L}^*_{\mathsf{Q}}$. Suppose that $(\Sigma_c)_{c \in C}$ is a family of maximally consistent (or irreflexive) theories of $\mathcal{L}^*_{\mathsf{Q}}$ with $\Sigma_c \; R_{\mathsf{N}} \; \Sigma_{c'}$ for all $c, c' \in C$. Then the set*

$$\Sigma^* := \bigcup_{c \in C} \{\sigma \in \mathcal{L}^*_{\mathsf{Q}} : \mathsf{Q}c\,\sigma \in \Sigma_c\}$$

is consistent.

Proof. (Cp., for example, Lemma 17-8 in Belnap et al. (2001), pp. 434 f.) Suppose that Σ^* is not consistent. Then there are $c_0, \ldots, c_n \in C$ and formulae $\sigma_0, \ldots, \sigma_n$ with $\mathsf{Q}c_j\,\sigma_j \in \Sigma_{c_j}$ for each $j \in \{0, \ldots, n\}$ such that

$$\sigma_0 \wedge \cdots \wedge \sigma_n \rightarrow \bot$$

is provable. It is obvious that in this case $n \geq 1$ must hold, since otherwise the contradiction could be proved from a single consistent theory. But then we obtain that

$$\sigma_0 \rightarrow (\neg\sigma_1 \vee \cdots \vee \neg\sigma_n)$$

is provable, and hence so is

$$\mathsf{Q}c_0\,\sigma_0 \rightarrow \mathsf{Q}c_0\,(\neg\sigma_1 \vee \cdots \vee \neg\sigma_n).$$

From $\mathsf{Q}c_0\,\sigma_0 \in \Sigma_{c_0}$ it follows that $\mathsf{Q}c_0\,(\neg\sigma_1 \vee \cdots \vee \neg\sigma_n) \in \Sigma_{c_0}$. Consequently, by (A-16), Σ_{c_0} contains the formula

$$\mathsf{N}\big(\mathsf{Q}c_1\,\sigma_1 \rightarrow \mathsf{N}\big(\mathsf{Q}c_2\,\sigma_2 \rightarrow \cdots \mathsf{N}(\mathsf{Q}c_{n-1}\,\sigma_{n-1} \rightarrow \mathsf{N}\neg\mathsf{Q}c_n\,\sigma_n) \cdots\big)\big).$$

Hence, by $\mathsf{Q}c_1\,\sigma_1 \in \Sigma_{c_1}$, Σ_{c_1} contains

$$\mathsf{N}\big(\mathsf{Q}c_2\,\sigma_2 \rightarrow \cdots \mathsf{N}(\mathsf{Q}c_{n-1}\,\sigma_{n-1} \rightarrow \mathsf{N}\neg\mathsf{Q}c_n\,\sigma_n) \cdots\big).$$

Continuing successively, we get that $\neg\mathsf{Q}c_n\,\sigma_n$ is in Σ_{c_n} – in contradiction to $\mathsf{Q}c_n\,\sigma_n \in \Sigma_{c_n}$. □

Thus, by this lemma, we have seen that Σ^* is consistent and hence extendable to a maximally consistent theory. But, in the context of the completeness proof presented here, this lemma does not help, since we have to show that there is an *irreflexive theory* in which Σ^* is included (provided the theories Σ_c of the lemma are irreflexive).

Proof of Lemma 6.10. Let C be a finite and Ω-disjoint set of individual constants. We can assume that $C = \{c_0, c_1, \ldots, c_n\}$ and that $c_i \not\supseteq^{\Omega} c_j$ for all $i \neq j$. Consider a map $\pi : C \to \Omega$ with $\Sigma_c := \pi(c)\ R_{\mathsf{N}}\ \pi(c')$ for all $c, c' \in C$. We have to show that there is an irreflexive theory $\Sigma^\pi \in \Omega$ with

$$\Sigma_c\ R_c\ \Sigma^\pi, \quad \text{for each } c \in C.$$

For each $c \in C$ choose a propositional constant q_c of $\mathcal{L}^*_{\mathsf{Q}}$ with $\chi_c := \chi_{q_c} := q_c \wedge \mathsf{N}^{\neq}\neg q_c \in \Sigma_c$. Then Σ_{c_0} contains the formula

$$(*) \qquad \neg \mathsf{Q}c_0 \neg(\neg \mathsf{Q}c_1 \neg \chi_{c_1} \wedge \cdots \wedge \neg \mathsf{Q}c_n \neg \chi_{c_n}).$$

Otherwise, the formula

$$\mathsf{Q}c_0 (\neg\neg \mathsf{Q}c_1 \neg \chi_{c_1} \vee \cdots \vee \neg\neg \mathsf{Q}c_n \neg \chi_{c_n})$$

and then, by (A-16),

$$\mathsf{N}\big(\mathsf{Q}c_1 \neg \mathsf{Q}c_1 \neg \chi_{c_1} \to \cdots \\ \mathsf{N}(\mathsf{Q}c_{n-1} \neg \mathsf{Q}c_{n-1} \neg \chi_{c_{n-1}} \to \mathsf{N}\neg \mathsf{Q}c_n \neg \mathsf{Q}c_n \neg \chi_{c_n}) \cdots\big)$$

would be elements of Σ_{c_0}. By $\mathsf{Q}c_j \neg \mathsf{Q}c_j \neg \chi_{c_j} \in \Sigma_{c_j}$, Σ_{c_n} would then contain $\neg \mathsf{Q}c_n \neg \mathsf{Q}c_n \neg \chi_{c_n}$ – in contradiction to $\mathsf{Q}c_n \neg \mathsf{Q}c_n \neg \chi_{c_n} \in \Sigma_{c_n}$.

Since $(*)$ holds and since Ω is closed, there exists an irreflexive theory Σ^π in Ω such that, for each $j \in \{1, \ldots, n\}$, $\Sigma_{c_0}\ R_{c_0}\ \Sigma^\pi$ and $\neg \mathsf{Q}c_j \neg \chi_{c_j} \in \Sigma^\pi$. For each $j \in \{1, \ldots, n\}$, by $\neg \mathsf{Q}c_j \neg \chi_{c_j} \in \Sigma^\pi$, there exists an irreflexive theory $\Sigma_j \in \Omega$ with $\Sigma^\pi\ R_{c_j}\ \Sigma_j$ and $\chi_{c_j} \in \Sigma_j$. By Lemma 6.3(b), we obtain that, for each $j \in \{1, \ldots, n\}$, $\Sigma_{c_j} = \Sigma_j$ and hence that, for each $j \in \{0, \ldots, n\}$, $\Sigma_{c_j}\ R_{c_j}\ \Sigma^\pi$. □

Now, consider again the agent-frame $\mathcal{F}^\Omega$ of Theorem 6.9. We define *the canonical* (*weak*) T×W-*based agent-structure*, $\mathcal{A}^\Omega$, on this frame by the following settings:

$$|c|^{\mathcal{A}^\Omega} := [\,c\,]_{\cong^\Omega},$$
$$|p|^{\mathcal{A}^\Omega} := \big\{(\Xi, \Psi) \in T^\Omega \times W^\Omega : p \in \Sigma_{\Xi,\Psi}\big\},$$

where c is an individual constant and p is a propositional constant of $\mathcal{L}^*_{\mathsf{Q}}$. The name of the structure so defined is justified by

THEOREM 6.12. *Let Ω be a maximally connected and weakly $\mathcal{L}^*_{\mathsf{Q}}$-independent set of irreflexive theories. Then, for each formula φ of $\mathcal{L}^*_{\mathsf{Q}}$, the equivalence*

$$\mathcal{A}^\Omega \models_{\Xi,\Psi} \varphi \iff \varphi \in \Sigma_{\Xi,\Psi}$$

holds at each pair $(\Xi, \Psi) \in T^\Omega \times W^\Omega$.

THEOREM 6.13 (Completeness of $\mathfrak{K}_{\mathsf{Q}}$). *The calculus* $\mathfrak{K}_{\mathsf{Q}}$ *is strongly complete with respect to weak* T×W*-based agent-structures, i.e., each consistent theory is satisfiable in a weak* T×W*-based agent-structure.*

Proof. Let Σ be a consistent theory of $\mathcal{L}_{\mathsf{Q}}$. Then, by Lemma 6.2, there is an irreflexive theory Σ^* of $\mathcal{L}^*_{\mathsf{Q}}$ in which Σ is included. The set of all irreflexive theories of $\mathcal{L}^*_{\mathsf{Q}}$ is closed (cf. Lemma 6.5) and hence, by Lemma 6.7, so is the maximally connected set of irreflexive theories, Ω_{Σ^*}, that contains Σ^*. By Lemma 6.10, Ω_{Σ^*} is weakly $\mathcal{L}^*_{\mathsf{Q}}$-independent and thus the frame $\mathcal{F}^{\Omega_{\Sigma^*}}$ of Theorem 6.9 is a weak T×W-based agent-frame. Consider the equivalence classes $\Xi_{\Sigma^*} \in T^{\Omega_{\Sigma^*}}$ and $\Psi_{\Sigma^*} \in W^{\Omega_{\Sigma^*}}$ to which Σ^* belongs. The canonical structure that is defined on the frame $\mathcal{F}^{\Omega_{\Sigma^*}}$ satisfies all formulae of Σ^* at time-point Ξ_{Σ^*} and world Ψ_{Σ^*} (cf. Theorem 6.12). From the canonical structure we obtain a structure of the language $\mathcal{L}_{\mathsf{Q}}$ by omitting all propositional constants of $\mathcal{L}^*_{\mathsf{Q}}$ that are new for $\mathcal{L}_{\mathsf{Q}}$. This structure satisfies Σ at Ξ_{Σ^*} and Ψ_{Σ^*}. □

As immediate corollaries we obtain:

COROLLARY 6.14 (Compactness of $\mathfrak{K}_{\mathsf{Q}}$). *A theory* Σ *is satisfiable in a weak* T×W*-based agent-structure if and only if each finite subset of* Σ *is satisfiable in a weak* T×W*-based agent-structure.*

THEOREM 6.15. *The calculus* $\mathfrak{K}_{\mathsf{Q}}$ *is weakly complete with respect to strong* T×W*-based agent-structures, i.e., each formula that is valid in every strong* T×W*-based agent-structure is provable in* $\mathfrak{K}_{\mathsf{Q}}$.

Proof. Use Theorem 6.13 and Theorem 4.2(b). □

Finally, modal connectivity can easily be axiomatized within the presented linguistic framework (cf. Wölfl (1999a)): Consider the axiom

(MC) $\quad \diamondsuit\varphi \to \mathsf{M}\varphi \vee \mathsf{PMF}\varphi.$

This axiom is valid in each structure defined on a modally connected T×W-frame (cf. Section 2). With regard to the completeness proof the concept of connectivity (cf. Definition 6.4) has to be redefined by substituting the relation R_{N} for $R_{\boxminus}$. Then (MC) guarantees that each maximally connected subset (in the new sense) of a closed set of theories is closed, too. It can be verified, then, that the canonical T×W-frame defined above is modally connected. Furthermore, by assessing the cardinality of the set of time-points and the set of worlds of this frame, we obtain:

THEOREM 6.16. *The calculus* $\mathfrak{K}_{\mathsf{Q}}$ + (MC) *is weakly complete with respect to the semantics of strong* T×W*-based agent-structures. Each formula that is satisfiable in a modally connected* T×W*-based agent-structure* *is satisfiable in a modally connected* T×W*-based agent-structure that is defined on a frame with a countable set of time-points and a countable set of worlds.*

Proof. Obviously, each $\Psi \in W^{\Omega}$ contains countably many irreflexive theories (since $\mathcal{L}^*_{\mathsf{Q}}$ has only denumerably many propositional constants). Then T^{Ω} is countable, because there is a bijection between each Ψ and T^{Ω}. Furthermore, by Proposition 6.8, we get that

$$\mathrm{card}(W^{\Omega}) \leq \mathrm{card}(T^{\Omega} \times W^{\Omega}) = \mathrm{card}(\Omega).$$

Let now $\Psi_0 \in W^{\Omega}$ be a fixed 'world' of the canonical frame. As the canonical frame is modally connected, it holds that

$$\Omega = \bigcup_{\Xi \in T^{\Omega}} \Omega_{\Xi,0},$$

where $\Omega_{\Xi,0} := \bigcup_{\Psi \sim_{\Xi} \Psi_0} \Psi$. Thus, since each $\Omega_{\Xi,0}$ is countable, so is Ω and hence W^{Ω}. □

7. Modal Divergence

As we have seen in Section 4, the semantics of tree-based agent-structures yields the same logic as the semantics of modally divergent and modally complete T×W-based agent-structures. Since it is still an open question how modal completeness can be axiomatized, we have to restrict our considerations to such T×W-frames in which the set of time-points has a maximal element. Each such frame is modally complete (cf. Proposition 2.3). The class of T×W-frames with ending time, i.e., the class of T×W-frames that have a last time-point $t_{\max}$, can be axiomatized by the formula

(Fin) $\quad \mathsf{G}\bot \vee \mathsf{FG}\bot.$

Note that, if this formula is added to $\mathfrak{K}_{\mathsf{Q}}$ as an axiom, the formula

$$\boxminus \mathsf{G}\bot \vee \boxminus \mathsf{FG}\bot$$

becomes provable. Thus, it remains the question whether modal divergence can be axiomatized within the given framework.

PROPOSITION 7.1. *The formula*

(MD) $\quad \boxminus \vartheta_p \wedge \chi_q \rightarrow \mathsf{N}^{\neq}\mathsf{FNP}(\vartheta_p \wedge \neg q),$

where $\vartheta_p := p \wedge \mathsf{G}\neg p \wedge \mathsf{H}\neg p$ *and* $\chi_q := q \wedge \mathsf{N}^{\neq}\neg q$*, is valid in each weak* T×W*-based agent-structure that is modally divergent and hence valid in each weak tree-based agent-structure.*

Proof. Let $\mathcal{A}$ be a modally divergent agent-structure and consider $t \in T$ and $w \in W$ with $\mathcal{A} \models_{t,w} \boxminus\vartheta_p \wedge \chi_q$. Let $w' \sim_t w$ with $w \neq w'$. From modal divergence of $\mathcal{A}$ it follows that there is a time-point t' with $w \not\sim_{t'} w'$. Thus, $t < t'$ must hold. Now, provided $w'' \sim_{t'} w'$, w and w'' must be distinct. Consequently, $\mathcal{A} \models_{t,w''} \vartheta_p \wedge \neg q$ and hence $\mathcal{A} \models_{t',w''} \mathsf{P}(\vartheta_p \wedge \neg q)$. □

Thus, the calculus $\mathfrak{K}_{\mathsf{Q}}$ + (MD) is sound with respect to tree-based agent-structures. But this axiom guarantees modal divergence of the canonical frame, too. Let $\Psi, \Psi' \in W^{\Omega}$ be distinct equivalence classes as in Section 6. We can assume that there is a $\Xi \in T^{\Omega}$ with $\Psi \sim^{\Omega}_{\Xi} \Psi'$ (i.e., $\Sigma_{\Xi,\Psi} R_{\mathsf{N}} \Sigma_{\Xi,\Psi'}$), or else Ψ and Ψ' are not $\sim^{\Omega}_{\Xi}$-related for each and hence for some $\Xi \in T^{\Omega}$. Let now p and q be propositional constants with $\boxminus\vartheta_p \wedge \chi_q \in \Sigma_{\Xi,\Psi}$ and hence, by (T-1), $\mathsf{GH}(\vartheta_p \to q) \in \Sigma_{\Xi,\Psi}$. By (MD), we get $\mathsf{N}^{\neq}\mathsf{FNP}(\vartheta_p \wedge \neg q) \in \Sigma_{\Xi,\Psi}$ and hence $\mathsf{FNP}(\vartheta_p \wedge \neg q) \in \Sigma_{\Xi,\Psi'}$, i.e., there is a $\Xi' >^{\Omega} \Xi$ with $\mathsf{NP}(\vartheta_p \wedge \neg q) \in \Sigma_{\Xi',\Psi'}$. Then it cannot hold that $\Sigma_{\Xi',\Psi'} R_{\mathsf{N}} \Sigma_{\Xi',\Psi}$; otherwise $\mathsf{P}(\vartheta_p \wedge \neg q)$ would be in $\Sigma_{\Xi',\Psi}$ – in contradiction to the fact that $\mathsf{H}(\vartheta_p \to q) \in \Sigma_{\Xi',\Psi}$. In summary, we obtain that there is a $\Xi' \in T^{\Omega}$ with $\Psi \not\sim^{\Omega}_{\Xi'} \Psi'$. Thus, the canonical T×W-based agent-structure is modally divergent. We can conclude:

THEOREM 7.2. *The calculus* $\mathfrak{K}_{\mathsf{Q}}$ + (Fin) + (MD) *is sound and weakly complete with respect to the semantics of tree-based agent-structures with ending time.*

8. SEEING TO IT THAT

In what follows, I will shortly discuss some applications of the completeness result presented in Section 6 to *stit*-logics, i.e., logical systems that axiomatize the various explications of the term 'seeing to it that'. We introduce new binary operators, which function syntactically like the Q-operator: astit (for 'achievement stit') and dstit (for 'deliberative stit'). With respect to T×W-based agent-structures these operators can be characterized semantically by:

$$\begin{aligned}
\mathcal{A} \models_{t,w} \mathsf{dstit}\, c\, \varphi \iff\ & \mathcal{A} \models_{t,w'} \varphi, \text{ for all } w' \in f_{|c|^{\mathcal{A}}}(t,w), \text{ and} \\
& \mathcal{A} \not\models_{t,w'} \varphi, \text{ for some } w' \sim_t w, \\
\mathcal{A} \models_{t,w} \mathsf{astit}\, c\, \varphi \iff\ & \text{there is a } t' < t \text{ such that} \\
& \mathcal{A} \models_{t,w'} \varphi, \text{ for all } w' \in f_{|c|^{\mathcal{A}}}(t',w), \text{ and} \\
& \mathcal{A} \not\models_{t,w'} \varphi, \text{ for some } w' \sim_{t'} w.
\end{aligned}$$

Now, both operators can easily be axiomatized by adding the following axioms to the calculus $\mathfrak{K}_{\mathsf{Q}}$:

(A-15) $\mathsf{dstit}\, c\, \varphi \leftrightarrow \mathsf{Q}c\, \varphi \wedge \mathsf{M}\neg\varphi$.
(A-16) $\boxminus\vartheta_p \to (\mathsf{astit}\, c\, \varphi \leftrightarrow \mathsf{P}(\mathsf{Q}c\, \mathsf{G}(\vartheta_p \to \varphi) \wedge \mathsf{MF}(\vartheta_p \wedge \neg\varphi)))$.

Axiom (A-15) defines dstit explicitly, while (A-16) gives a merely partial or 'local' definition for astit (like local definitions for the Since- and the Until-operator in ordinary temporal logic). Indeed, (A-16) presents a formula that axiomatizes astit completely (each irreflexive theory contains a formula of the form $\boxminus\vartheta_p$). In particular, the axioms for astit discussed in Xu (1995b) can be proven within this calculus. Furthermore, we obtain the following theorem:

$$\boxminus\vartheta_p \wedge \mathsf{astit}\, c\, \varphi \to \mathsf{P}\big(\mathsf{dstit}\, c\, \mathsf{F}(\vartheta_p \wedge \varphi) \wedge \mathsf{G}\neg\mathsf{dstit}\, c\, \mathsf{F}(\vartheta_p \wedge \varphi)\big).$$

This theorem can be considered as a formula expressing: If an agent *a has* seen to it that φ holds now, there exists exactly one time-point in the past at which *a* sees to it that φ *will* hold now.

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NOTES

[1] Note, however, that $\mathscr{B}^{(\mathscr{R}^{\mathscr{B}})}$ and $\mathscr{B}$ do not coincide, but that they are isomorphic to each other. An analogous claim holds for $\mathscr{R}^{(\mathscr{B}^{\mathscr{R}})}$ and $\mathscr{R}$.

[2] But cp. Reynolds (2001).

[3] Note that there are formulae that, in general, are not true in T×W-frames, but are true in modally complete frames (cf., for example, Thomason (1984)).

[4] Cf. Wölfl (1999a).

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