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Events in Branching Time

Abstract. The concept of event is one of the key notions of many theories dealing with causality or agency. In this paper we study different approaches to events that share the basic assumption that events can be analyzed fruitfully in branching-time structures. The terminological framework developed thereby may be helpful for further analyses in the fields of causality and agency and also in those fields of computational semantics, where similar concepts are considered.

Keywords: Event, state, tree, causality, transition

1. Introduction

The category of events is the subject of a perpetual and sometimes controversial philosophical debate. From an ontological point of view, it is even contentious whether events form a category in their own right. Many authors have argued that events are reducible to, or supervene on, ontologically less questionable entities such as objects, propositions, or properties.[1] But obviously, reducibility considerations rely crucially on more or less precise definitions of the concepts involved and hence on criteria that characterize an entity as an event, a proposition, etc. On the other hand a discussion of the defining criteria for events and related concepts can yield fruitful results only if these notions are embedded into an ontological background theory. Making explicit the background assumptions may help clarify the ontological commitments on which these notions depend.

The concept of event plays an important role in very different research areas. But looking at how the usage of the term "event" varies with its context, it appears at least arguable whether there is a unique concept of event that is applicable in all contexts dealing with event-like entities. To put it another way: the common core of the different usages of "event" may be too small to provide a definition of events that suffices to distinguish them from other ontological entities.

In this paper we aim at developing a concept of event that is especially suited for analyses of causal and action-theoretical notions. More precisely, we study various concepts that are defined in terms of branching-time, tree-

[1] For a short summary of the discussions see Casati and Varzi [5].

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like models. These models are a natural choice for representing situations with indeterministic aspects. Belnap, Perloff, and Xu [3], for example, used branching-time structures for modeling an agent's choices to act in an indeterministic world. The assumption of indeterminism is fundamental for that approach: By acting a person may influence the future evolution of the world, i. e., the future flow of events cannot be settled completely, when the agent intervenes. Actions, however, may count as paradigmatic examples of events.

This paper presents a formal theory of events that provides an explicit definition of the concept of event. A different approach would have been to specify events implicitly by listing formal (algebraic or mereological) properties that hold between events. The main reason for preferring an explicit to an implicit approach is that these formal properties are far from being clear, and hence topic of the formal theory discussed in the following. On the other hand, the main focus of the paper entails that many philosophical topics related to events are to be disregarded in what follows. In particular, we will not discuss the relationship between the presented concept of event and typical ontological notions such as "change", "property instantiation", "trope", "particular", and "substance". Furthermore, we restrict consideration to *singular events*, i. e., to events that are not repeatable.

In natural languages one finds different means for referring to events. Events can be referred to indirectly by verb tenses (e. g., by progressive verb tenses as in "Tim is opening the window"), or directly via nominal phrases ("the murder of Julius Caesar") or proper names ("the Big Bang", "the Interregnum"). But we do not stick too much to the linguistic surface of our ordinary talk about events. It seems that this event talk reflects pragmatic issues rather than ontological insights (cp., e. g., Bennett [4]). For example, in some contexts the terms "John's pleasant walk" and "John's walk" may be used to refer to the same event, while in other contexts it may not. We take the view that whether two such terms denote the same event presents an interesting semantical question, but not an ontological problem.

Sometimes events are contrasted with *processes* to emphasize that events are "small" entities. We do not adopt this terminology, but rather subsume the notion of process under the notion of event. Often the notion of process is employed to stress some regulatory aspects of events. A detailed discussion of such events will not be carried out here, but some of the observations presented in the last three sections of the paper sketch the lines along which a subtler notion of process (as an event controlled by causal laws) could be defined.

The paper is organized as follows: In section 2 we first recall some basic concepts of the theory of branching-time structures. Then in sections 3 and 4 we develop a concept of event by generalizing step-by-step a definition proposed by Kutschera [10]. In section 5 we discuss some mereological and algebraic properties that could be used to characterize events implicitly. Section 6 discusses the relationship between events and propositions. In sections 7–9 we compare our concept of event to other approaches discussed in the literature. In particular, we show how events correspond to *transitions*, a notion discussed by Xu [32], and we consider a concept of event developed by Meixner [16].

2. Branching-time Structures

To begin with, let us recall some basic concepts of the theory of branching-time structures.[2] The most fundamental notion, of course, is that of a *tree*: A *tree* $\mathscr{T}$ is defined by a non-void set of *moments*, Mom, and a binary relation, $\prec$, by which Mom is partially ordered (in the strict sense); that is, the relation $\prec$ is irreflexive and transitive. Furthermore, we assume that the relation $\prec$ does not allow for *backward branching*, i. e., $\prec$ is linear-to-the-left[3] on Mom. We read "$m \prec m'$" as "m is (causally) earlier than m'".

Symbols like $\preccurlyeq$, $\succ$, and $\succcurlyeq$ are used in the natural manner. For sets M and M' of moments, we write $M \preccurlyeq M'$ ($M \prec M'$) if for all $m \in M$ and $m' \in M'$, it holds that $m \preccurlyeq m'$ ($m \prec m'$). Finally, $M \preccurlyeq m$ is an abbreviation of $M \preccurlyeq \{m\}$, etc.

A *chain* in a tree $\mathscr{T}$ is a set of moments that is linearly ordered by the relation of earlier-than. Note that, by Zorn's Lemma, each chain of moments is included in a maximal chain. A chain is said to be *trivial* if it contains less than two elements. A chain k is *upper-bounded* if there exists a moment m such that $k \preccurlyeq m$. In an analogous manner concepts such as *lower-bounded* or *bounded* are introduced.

Maximal chains in a tree $\mathscr{T}$ are said to be *histories*. Variables h, h', $h_1, \ldots$ will be used to denote histories, and His is defined as the set of all histories. For a given moment m, let $\mathrm{His}\langle m\rangle$ be the set of all histories that *pass through* m, i. e., $\mathrm{His}\langle m\rangle$ is the set of all maximal chains that contain m as element. Furthermore, we will use the following notions: Histories h and h' are *connected* if they intersect; h and h' are *undivided* at moment m,

[2] Here and in what follows we will widely adopt notions and notations proposed by Belnap et al. [3].

[3] This means that, for all moments m and m', if there exists a moment m'' such that $m, m' \prec m''$, then either $m \prec m'$ or $m = m'$, or $m' \prec m$.

symb. $h \curlywedge_m h'$, if there exists a moment $m' \in \text{Mom}$ such that $m' \succ m$ and $m' \in h \cap h'$; h and h' *split* at moment m, symb. $h \perp_m h'$, if m is the maximal element of $h \cap h'$;[4] h and h' are *separated* at moment m if either $m \in h \setminus h'$ or $m \in h' \setminus h$.

A *synchronized tree* is defined by a set of moments, Mom, a linear-to-the-left partial order on Mom, $\prec$, and a partition Inst of Mom. The elements of Inst are referred to as *instants*. We require that synchronized trees satisfy the following conditions:

(a) For each $h \in \text{His}$ and each $i \in \text{Inst}$, there exists exactly one element $m_{i,h} \in i \cap h$.
(b) For all histories $h, h' \in \text{His}$, the assignment $m_{i,h} \mapsto m_{i,h'}$ preserves the linear orders defined by $\prec$ on h and h' respectively.

For each moment m, let i_m be the unique element of Inst that contains m as element. Moment i_m is referred to as *the instant of moment* m.

It is worth noting that, in general, the partition Inst is not uniquely determined by conditions (a) and (b). The concepts of undividedness and splitting can be reformulated with respect to synchronized trees as follows: Histories h and h' are *undivided* at instant i, symb. $h \curlywedge_i h'$, if they are undivided at moment $m_{i,h}$ (or equivalently, if h and h' are undivided at moment $m_{i,h'}$). Histories h and h' *split* at instant i if they split at moment $m_{i,h}$. Note that each history h of a synchronized tree induces a map $\hat{h}$: Inst $\to$ Mom that assigns to each instant i the unique element $m_{i,h}$ contained in $i \cap h$. Since this map is bijective, the relation $\prec$ defines a linear order on Inst by:

$$\begin{aligned} i < i' :\Longleftrightarrow{}& m_{i,h} \prec m_{i',h}, \text{ for some history } h \\ \Longleftrightarrow{}& m_{i,h} \prec m_{i',h}, \text{ for each history } h. \end{aligned}$$

Then for each history h, the map $\hat{h}$ is an isomorphism with respect to the linear orders induced by $\prec$ on Inst and h respectively.

Definition 2.1. Let $\mathscr{T}$ be a (synchronized) tree.

(a) $\mathscr{T}$ is said to be *indeterministic* if there exist distinct histories $h, h' \in$ His that are connected (i. e., h and h' intersect). Otherwise, $\mathscr{T}$ is said to be *deterministic*.
(b) $\mathscr{T}$ is said to be *connected* if each pair of histories of $\mathscr{T}$ is connected, i. e., if for each pair of moments $m, m' \in \text{Mom}$, there exists a moment m'' such that $m'' \preccurlyeq m$ and $m'' \preccurlyeq m'$.

[4] Note that the intersection of a pair of histories need not have a maximal element (even if they intersect).

(c) $\mathscr{T}$ is said to be *dense* if $\prec$ is dense in Mom, i. e., if for each pair of moments $m \prec m'$, there exists a moment m'' such that $m \prec m'' \prec m'$.

(d) $\mathscr{T}$ is said to be *sup-closed* if for each non-void and upper-bounded chain k in $\mathscr{T}$ and each history $h \in$ His that contains k as a subset, there exists a (uniquely determined) least upper bound of k in h, symb. $\sup_h k$, namely the *supremum* of k in h (i. e., $k \preccurlyeq \sup_h k \in h$ and $\sup_h k \preccurlyeq m$ for each moment m with $k \preccurlyeq m \in h$).

(e) $\mathscr{T}$ is said to be of *splitting type 1* if for each pair of distinct but connected histories $h, h' \in$ His, the intersection of h and h' has a maximum (i. e., there exists a moment at which h and h' split).[5]

(f) $\mathscr{T}$ is said to be of *splitting type 2* if for each pair of distinct histories h and h', the intersection of h and h' has no maximum (i. e., provided h and h' are connected, h and h' are undivided at each moment $m \in h \cap h'$).

(g) $\mathscr{T}$ is said to be of *splitting type 3* if for each pair of distinct but connected histories h and h', the chain $h \setminus h'$ has no minimum.

(h) $\mathscr{T}$ is said to be of *splitting type 4* if for each pair of distinct but connected histories h and h', the chain $h \setminus h'$ has a minimum.

To illustrate these concepts we note some propositions:

PROPOSITION 2.2. *Let $\mathscr{T}$ be a sup-closed synchronized tree and let k be a non-void and upper-bounded chain in $\mathscr{T}$. Then the suprema of k in all histories containing k are simultaneous. More precisely, if $k \subseteq h$ and $k \subseteq h'$, then $i_{\sup_h k} = i_{\sup_{h'} k}$.* ■

PROPOSITION 2.3. *Each sup-closed synchronized tree is inf-closed; that is, each non-void and lower-bounded chain k has a greatest lower bound $\inf k$ (i. e., $\inf k \preccurlyeq k$ and for each $m \preccurlyeq k$, $m \preccurlyeq \inf k$).*

PROOF. Consider the upper-bounded chain $k^* := \{\, m : m \preccurlyeq k \}$. Let h be a history that contains k as subset. Obviously, k^* is a non-void and upper-bounded subset of h. Hence $\sup_h k^*$ exists and is the greatest lower bound of k. ■

PROPOSITION 2.4. (a) *Each connected and deterministic tree has exactly one history.*

(b) *There is no indeterministic and dense tree that is both of splitting type 1 and of splitting type 4.*

(c) *Each dense and sup-closed tree is of splitting type 1 if and only if it is of splitting type 3.*

[5] In the literature this condition is also known as *semi-lattice* condition.

(d) *Each dense and sup-closed tree is of splitting type 2 if and only if it is of splitting type 4.* ■

3. Generalized Kutschera Events

The concept of event presented in the sequel aims at generalizing a definition proposed by Kutschera [9]. The basic idea of Kutschera's approach can be briefly illustrated as follows: If an event, say the murder of Caesar, occurs, then it takes place within a period of time. But the murder of Caesar could have started earlier (or later) than it actually did. Moreover, it could have been the case this event did not take place at all. Obviously the murder of Caesar is a singular event, since it can take place only once. Hence it is suggesting to represent an event, such as the murder of Caesar, via its temporal traces in the various histories in which it happens. Consequently, events E and E' are distinct if there exists a history in which only one of them takes place or if there is a history in which both take place, but within different periods of time.

To present Kutschera's approach in more detail, let us consider a synchronized tree $\mathscr{T}$. A *closed history segment* is a chain of moments, k, that can be written in the form $k = \hat{h}(\tau) := \{\, \hat{h}(i) \,:\, i \in \tau \}= \{\, m_{i,h} \,:\, i \in \tau \}$ for some history h and some closed bounded interval τ of instants.[6]

Definition 3.1 (cf. Kutschera [10]). Let $\mathscr{T}$ be a synchronized tree. A *K-event* is a non-void set of closed history segments of $\mathscr{T}$, E, that meets the following conditions:

(a) From $\hat{h}(\tau), \hat{h}(\tau') \in E$ it follows that $\tau = \tau'$.
(b) If $\hat{h}(\tau), \hat{h'}(\tau') \in E$ and $\hat{h}(\tau) \cap \hat{h'}(\tau') \neq \emptyset$, then $\min \tau = \min \tau'$.

The first of these conditions expresses that events are understood as singular events, that is, that an event occurs in a history only once provided it occurs in that history at all. To explain the second condition, let us assume that an event E occurs in two distinct histories h and h' starting in moments $m \in h$ and $m' \in h'$ respectively. Now if both moments m and m' are contained in $h \cap h'$, condition (b) enforces that E starts in both histories at the same moment, i. e., m and m' must be identical. For example, if a person α

[6] A closed (bounded) interval of instants is a set τ of instants that can be written as $\tau = [i_1, i_2] := \{\, i \in \mathrm{Inst} \,:\, i_1 \preccurlyeq i \preccurlyeq i_2 \}$ for distinct instants i_1 and i_2; an open (bounded) interval is a set τ of instants of the form $\tau = (i_1, i_2) := \{\, i \in \mathrm{Inst} \,:\, i_1 \prec i \prec i_2 \}$. Note that in these cases $\hat{h}(\tau)$ is a closed (or open) interval with respect to the linear order defined by $\prec$ on history h.

is just driving from a location A to a location B and now (at moment m) she can decide whether to go to B via a location C or via a location C', we have (at least) two histories in which her driving from A to B occurs: one in which she goes via C and one in which she goes via C'. In both histories α's driving from A to B obviously starts at the same moment. However, *before* actually starting to drive, α could have decided to start later than she did. Hence if histories h and h' split before α starts to drive in h, then α's driving in h' may start later than her driving in h. A representation of this situation is depicted in Figure 1. Let E be α's driving from A to B. Let

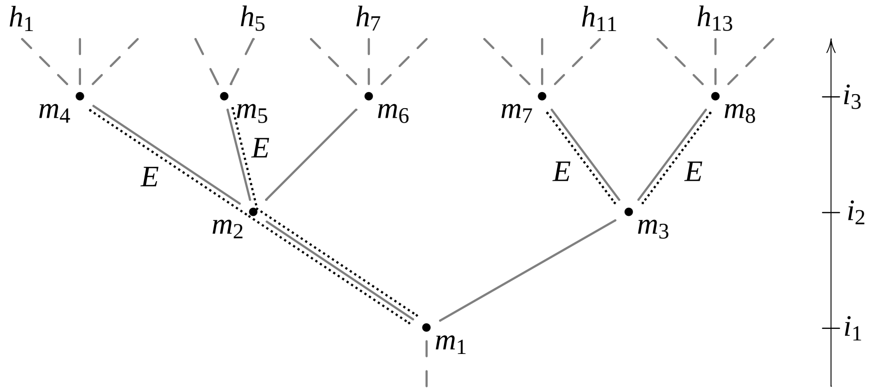


Figure 1. A K-event in a synchronized tree. E is defined by $\{h_1[i_1, i_3], h_5[i_1, i_3], h_{11}[i_2, i_3], h_{13}[i_2, i_3]\}$.

us assume that in histories h_1–h_3 α goes via C, and in h_4–h_5 she goes via C'. In histories h_9–h_{14}, α's driving from A to B starts later than her driving in histories h_1–h_5. In histories h_6–h_8 person α, tired of a series of traffic congestions, decides at moment m_2 to return to A, i. e., in these histories α's driving from A *to* B does not occur.

It is obvious that many incidents can be represented as K-events (for example, Tim's first chess-match against Tom, Mary's opening her office window this morning, an apple's fall from the tree on which it has grown). But there is also a large class of happenings that cannot count as events in the strict sense of Definition 3.1. Examples of such happenings include the flight of an arrow (understood as the arrow being in movement, not in rest) and a football game (not including its break). Hence, a first natural step for generalizing Kutschera's approach to events consists in dropping the requirement that events be sets of *bounded intervals* of moments.

Definition 3.2 (*Generalized K-event*). Let $\mathscr{T}$ be a tree. A *generalized K-event* in $\mathscr{T}$ is a non-void set of non-void chains in $\mathscr{T}$, E, such that the following *coincidence condition* is satisfied:

(CC) Let $e, e' \in E$ be chains, and let $h, h' \in \text{His}$ be histories such that $e \subseteq h$ and $e' \subseteq h'$. If $e \cap h'$ and $e' \cap h$ are non-void, then both sets are identical.

A generalized K-event E is said to *occur* in history h if E contains a chain e such that $e \subseteq h$. Provided m is a moment of history h, we say that E *runs at* m *in* h if there exists a chain $e \in E$ with $m \in e \subseteq h$. The set of histories in which E occurs is denoted by $\text{His}\langle E\rangle$.

For the sake of brevity, generalized K-events will be referred to as *events* in the sequel of this section. Then condition (CC) means that, if an event E occurs in histories h and h' and if these occurrences of E pass through a common part of both histories, then they pass through exactly the same moments until both histories are separated. Due to (CC), events are understood as *singular events*. This is expressed by the following lemma:

Lemma 3.3. *Let E be an event in a tree $\mathscr{T}$. Then for each history $h \in \text{His}\langle E\rangle$, there exists exactly one chain $e_{E,h} \in E$ such that $e_{E,h} \subseteq h$.*

Proof. This is an immediate consequence of (CC). ■

Lemma 3.3 corresponds exactly to condition 3.1(a). But, as we will see shortly, condition (CC) expresses more, since it also generalizes condition 3.1(b). Note that (CC) is stronger than other conditions that work as generalizations of 3.1(b), such as:

(CC') Let $e, e' \in E$ be chains in $\mathscr{T}$ with $e \cap e' \neq \emptyset$, and let $h, h' \in \text{His}$ be histories such that $e \subseteq h$ and $e' \subseteq h'$. Then $e \cap h'$ coincides with $e' \cap h$.

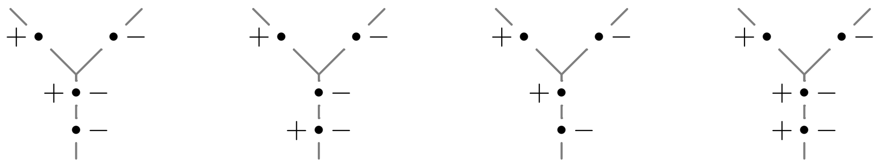
(CC") Let $e, e' \in E$ be chains in $\mathscr{T}$ and let m be a moment contained in $e \cap e'$. Then for each $m' \prec m$, $m' \in e$ if and only if $m' \in e'$.

Some of the differences between these conditions are illustrated in Figure 2. In each diagram depicted there an "event" $E_{+-} = \{e_+, e_-\}$ is defined by two chains of moments, namely e_+ consisting of the moments labeled with "+" and e_- consisting of the moments labeled with "−". The set of chains shown in Figure 2(c), for example, is not an event in the sense of Definition 3.2, since it fails to satisfy condition (CC).

However, all these conditions are equivalent with respect to *convex* events (cf. Lemma 3.9):

Definition 3.4. Let E be an event in $\mathscr{T}$.

(a) E is said to be *convex* if each $e \in E$ is convex with respect to $\prec$ (i. e., if $m', m'' \in e$ and $m' \prec m \prec m''$, then $m \in e$). Events that are not convex are said to be *scattered*.



(a) E_{+-} violates (CC), (CC'), and (CC").

(b) E_{+-} violates (CC) and (CC'), but satisfies (CC").

(c) E_{+-} violates (CC), but satisfies (CC') and (CC").

(d) E_{+-} satisfies (CC), (CC'), and (CC").

Figure 2. Condition (CC) and some of its variants. Let E_{+-} be the "event" that runs in the left history at the moments labeled with "+" and runs in the right history at the moments labeled with "−".

(b) E is said to be *distended* if each $e \in E$ is a non-trivial chain.

(c) E is said to be *finite* if each $e \in E$ has both a minimum, $\min e$, and a maximum, $\max e$, with respect to $\prec$ (i. e., $\min e, \max e \in e$ and $\min e \preccurlyeq e \preccurlyeq \max e$).

(d) E is said to be *bounded* if each $e \in E$ is bounded (i. e., there exist moments m and m' such that $m \preccurlyeq e \preccurlyeq m'$).

(e) E is said to be *progressive* if for each history $h \in \mathrm{His}\langle E\rangle$ and each moment $m \in e_{E,h}$, there exist moments $m_1, m_2 \in h$ such that $m_1 \prec m \prec m_2$ and $m' \in e_{E,h}$ for each m' with $m_1 \prec m' \prec m_2$.

(f) An event E is said to be *separated* if for all histories $h, h' \in \mathrm{His}\langle E\rangle$ with $\emptyset \neq e_{E,h} \cap h' \prec e_{E,h'}$, there exists a moment $m' \in h' \setminus h$ such that $e_{E,h} \cap h' \prec m' \preccurlyeq e_{E,h'}$. Otherwise, E is said to be *non-separated*.

Obviously, each finite event is bounded, and in trees of splitting type 3 each bounded event is separated. In dense trees finite events are never progressive, and progressive events are distended. The discussion of some examples may illustrate these new notions:

Example 3.5. Each non-void set H of histories of a tree $\mathscr{T}$ is an event that occurs exactly in the histories contained in H. The set $\{\mathrm{Mom}\}$ is an event if and only if Mom is linearly ordered by $\prec$, i. e., if $\mathscr{T}$ has exactly one history.

Example 3.6. For each non-void chain k in a tree $\mathscr{T}$, the singleton set $E_k := \{k\}$ defines an event in $\mathscr{T}$. This event occurs in all histories containing k as a subset. Furthermore, event E_k occurs in exactly one history if k has no upper bound. It is worthwhile to remark that in Belnap et al. [3] the elements of Mom are sometimes referred to as events. In fact, each moment m defines an event, namely $E_m := \{\{m\}\}$. Obviously, the *momentary event* E_m occurs in exactly those histories that pass through moment m, i. e., $\mathrm{His}\langle E_m\rangle = \mathrm{His}\langle m\rangle$.

Example 3.7. Let $\mathscr{T}$ be a synchronized tree. Consider an interval τ of instants and a non-void set H of histories of $\mathscr{T}$. Then

$$E_{H,\tau} := \{\ \hat{h}(\tau)\ :\ h \in H\ \}$$

is a convex event in $\mathscr{T}$ such that $H \subseteq \mathrm{His}\langle E_{H,\tau}\rangle$. Obviously, if τ is a closed (open) bounded interval of instants, then $E_{H,\tau}$ is finite (progressive).

Example 3.8. Let $\mathscr{T}$ be a dense tree of splitting type 1. Consider distinct, but connected histories h and h' of $\mathscr{T}$ that split at moment m. Then $\{h, h' \setminus h\}$ and $\{(h \setminus h') \cup \{m\}, h' \setminus h\}$ are non-separated events, while $\{h \setminus h', h' \setminus h\}$ is separated. Non-separated events are also definable in case that the tree at hand is not of splitting type 1. Consider, for example, a tree in which each history is isomorphic to $\mathbb{Q}$. Furthermore, assume that there are histories h and h' that are undivided at each moment (temporally) earlier than $\sqrt{2}$, but are separated at each later moment. Obviously, this tree is not of splitting type 1, and $\{h, h' \setminus h\}$ is an event that is not separated.

Lemma 3.9. *Let $\mathscr{T}$ be a tree and let E be a non-void set of convex and non-void chains in $\mathscr{T}$ such that each history contains at most one chain of E. Equivalent are:*

(i) *E is an event.*

(ii) *Let $e, e' \in E$ with $e \cap e' \neq \emptyset$ and let $h, h' \in \mathrm{His}$ with $e \subseteq h$ and $e' \subseteq h'$. Then $e \cap h'$ coincides with $e' \cap h$.*

(iii) *Let $e, e' \in E$ and $m \in e \cap e'$. Then for each $m' \prec m$, moment m' is an element of e if and only if m' is an element of e'.*

Proof. It is easy to see that (ii) follows from (i), and that (iii) follows from (ii). Hence, we just have to show that (i) follows from (iii). To prove this, let e, e' be chosen in E and let h and h' be histories such that $e \subseteq h$, $e' \subseteq h'$, $e \cap h' \neq \emptyset$, and $e' \cap h \neq \emptyset$. It suffices to show that $e \cap h'$ is contained in e'. Without restriction, we may assume that neither e is a subset of h' nor e' is a subset of h. Otherwise, both chains e and e' are contained in the same history and hence identical; in this case the claim holds trivially. Thus, we can fix moments $m_1 \in e \setminus h'$, $m'_1 \in e' \setminus h$, $m_2 \in e \cap h'$, and $m'_2 \in e' \cap h$. Let now m be chosen arbitrarily in $e \cap\ h'$. Then it follows that $m \prec m_1$ and $m \prec m'_1$. Furthermore, m, m_2 and m'_2 are comparable with respect to $\prec$, since each of them is contained in the chain $h \cap h'$. Now, if $m'_2 \preccurlyeq m$, then m is an element of e', since m'_2 and m'_1 are elements of the convex chain e'. And if $m \prec m'_2$, then $m \prec m'_2 \prec m_1$ and hence, by convexity, $m'_2 \in e$. Consequently, it holds that $m'_2 \in e \cap e'$ and hence, by (iii), that $m \in e'$. ■

Proposition 3.10. *Let $\mathscr{T}$ be a synchronized tree and let E be a set of chains in $\mathscr{T}$. Then the following statements are equivalent:*

(i) *E is a K-event in $\mathscr{T}$.*

(ii) *E is a distended, convex, and finite event in $\mathscr{T}$.*

Proof. First, let E be a distended, convex, and finite event in $\mathscr{T}$, and set $\tau_e := [i_{\min e}, i_{\max e}]$ for each $e \in E$. Then $i_m \in \tau_e$ for each $m \in e$, and hence $e = \hat{h}(\tau_e)$ for each history h containing e. Since E is distended, E is a set of closed history segments. Then, by Lemma 3.3, condition 3.1(a) is satisfied. Let now $e = \hat{h}(\tau)$ and $e' = \hat{h}'(\tau')$ be chains of E such that $\hat{h}(\tau)$ intersects with $\hat{h}'(\tau')$. Choose $m \in e \cap e'$. Then, by condition (CC), $e \cap h' = e' \cap h$ and hence, by $\min e \preccurlyeq m \in h'$, $\min e$ is contained in e'. Analogously, it can be shown that $\min e'$ is contained in e. Thus, $\min e' \preccurlyeq \min e$ and $\min e \preccurlyeq \min e'$, consequently $\min e = \min e'$, and finally $\min \tau = \min \tau'$.

For the other direction, let E be a K-event. Then for each $e \in E$, there exists a unique closed interval of instants, τ_e, such that for each history h with $e \subseteq h$, $e = \hat{h}(\tau_e)$. First, assume that there are chains $e, e' \in E$ and a history h with $e, e' \subseteq h$. Then, by applying condition 3.1(a) to $e = \hat{h}(\tau_e)$ and $e' = \hat{h}(\tau_{e'})$, it follows that $\tau_e = \tau_e'$ and hence that $e = e'$. Consequently, E satisfies the premise of Lemma 3.9 and condition 3.9(iii). For if $m' \prec m$, $m' \in \hat{h}(\tau)$, and $m \in \hat{h}(\tau) \cap \hat{h}'(\tau')$, then the minima of τ and τ' must be identical (cf. Definition 3.1(b)) and hence so are $\hat{h}(\min \tau)$ and $\hat{h}'(\min \tau')$ (note that the intersection of $\hat{h}(\tau)$ and $\hat{h}'(\tau')$ is non-void). Hence $\min \tau \preccurlyeq i_{m'} \prec i_m$ and consequently $m' \in \hat{h}'(\tau')$. Thus, E is an event in the sense of Definition 3.2. Obviously, E is distended, convex, and finite. ■

In summary, K-events are exactly those events that are distended, convex, and finite. Moreover, in section 5 we prove that, given certain conditions, each distended and bounded event can be embedded into a K-event.

4. A Refined Concept of Event

Summarizing the previous section, we can say that K-events are natural special cases of events in the sense of Definition 3.2. But is the concept of event provided by that definition wide enough? To discuss this in more detail, let us consider the event of Ahab's sailing in search of the White Whale — the event described in Melville's novel *Moby Dick*.[7] It is easy to find circumstances to the effect that Ahab's sailing in search of the White Whale can not count as a K-event (consider, for example, histories in which

[7] This example has been discussed by Belnap et al. [3] in a different context.

Ahab interrupts his search in order to make some repairs to his ship). But Ahab's sailing in search of the White Whale is not an event in the sense of Definition 3.2 either. To see this, let us assume that Ahab starts his sailing in history h_0 at moment m_0 and let us suppose that, after some days on the sea, say at moment $m_1 \in h_0$, Ahab thinks about giving up his search. Suppose that in history h_0 he continues to sail *in search of* the White Whale (until some later moment m_2), but that in history h_1 he gives up and returns home. Thus, it is quite natural to say that in history h_1 his sailing in search of the White Whale starts at moment m_0 and ends in m_1, while in h_0 this event occurs between m_0 and m_2. But then condition (CC) is obviously violated.

To put it another way: If Ahab's sailing in search of the White Whale were an event in the sense of 3.2, then Ahab could never give up his search after he started it. For if we assume that Ahab decided in h_1 at m_1 to give up his search (while in h_0 he proceeds), then there exist two ways of representing this situation: (a) In h_1 Ahab's sailing in search of the White Whale does not take place — this alternative is obviously unacceptable — or (b) Ahab's sailing occurs in h_1 and, since this occurrence started in h_1 at m_0, it is to end in h_1 at some moment m_3 later than m_2 (this is guaranteed by (CC); note that h_0 and h_1 are separated at m_1). Let us now consider some moment m_4 properly between m_2 and m_3. Should we then say that Ahab is still sailing in search of the White Whale, or not? If he were, this would contradict the assumption that Ahab gave up at some moment before. And if he were not, this would contradict that his sailing in search of the White Whale ends at moment m_3, which is later than m_4. Thus strategy (b) does not provide a satisfactory solution for the problem either.

The problem just discussed typically occurs where incidents are considered that cannot count as *success events*. A success event is an event that occurs only if also specific *immediate* (causal) effects of that event — certain states or events — are realized too. This means that these effects are considered as components of the event that have to be realized in order to realize the event as a whole: for example, the murder of Julius Caesar (Julius Caesar *is* dead after the murder), the fall of an apple from its tree (the apple lies on the earth and is no longer spatially connected with the tree), Ahab's finding the White Whale (Ahab sees the White Whale after having searched for it).

For this reason, it makes sense to generalize the concept of event introduced in the previous section as follows:

Definition 4.1 (*Event*). An *event* in $\mathscr{T}$ is a partial function ε with non-void domain $\operatorname{dom}\varepsilon \subseteq \text{His}$ such that for all histories $h, h' \in \text{His}$, the following conditions are satisfied:

(a) If $h \in \mathrm{dom}\,\varepsilon$, then $\emptyset \neq \varepsilon(h) \subseteq h$.
(b) If $h \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) \subseteq h'$, then $h' \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h') \cap h \neq \emptyset$.
(c) If $h, h' \in \mathrm{dom}\,\varepsilon$ with $\varepsilon(h) \cap h' \neq \emptyset$ and $\varepsilon(h') \cap h \neq \emptyset$, then $\varepsilon(h) \cap h' = \varepsilon(h') \cap h$.

ε is said to *occur* in history h if $h \in \mathrm{dom}\,\varepsilon$, and ε is said to *run* at moment m in history h if $h \in \mathrm{dom}\,\varepsilon$ and $m \in \varepsilon(h)$.

Obviously, events in the sense of this definition are singular events. If an event ε occurs in history h then it runs at some moments in that history and each moment at which event ε runs in history h is contained in h. This is expressed by condition (a). Condition (c) corresponds to the coincidence condition of Definition 3.2. Finally, condition (b) enforces that, if event ε occurs in a history h and if this occurrence is completely contained in a history h', then ε also occurs in h'. Note that in the presence of condition (c), condition (b) is equivalent to each of the following conditions:

(b') If $h \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) \subseteq h'$, then $h' \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) \subseteq \varepsilon(h')$.
(b') If $h \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) \subseteq h'$, then $h' \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) = \varepsilon(h') \cap h$.

On the other hand condition (b) is stronger than condition (b*):

(b*) If $h \in \mathrm{dom}\,\varepsilon$ and if there is a moment $m \in h \cap h'$ such that $\varepsilon(h) \prec m$, then $h' \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) \subseteq h'$.

There is only weak evidence for preferring condition (b) to condition (b*): it seems that choosing one of these conditions is just a matter of taste.[8]

Generalized K-events correspond one-to-one to events in the new sense that fulfill the following condition:

(d) If $h \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) \subseteq h'$, then $\varepsilon(h') \subseteq h$.

To see this, let E be a generalized K-event and define ε by $\mathrm{dom}\,\varepsilon := \mathrm{His}\langle E\rangle$ and $\varepsilon(h) := e_{E,h}$. Then it is easy to verify that each of the conditions (a)–(d) is satisfied. Vice versa, let ε be an event that meets condition (d). Let $h, h' \in \mathrm{dom}\,\varepsilon$ be histories with $\varepsilon(h) \subseteq h'$. By (d), we obtain that $\varepsilon(h')$ is contained in h and hence that $\varepsilon(h) = \varepsilon(h') \cap h = \varepsilon(h')$. Then $E := \{\,\varepsilon(h) : h \in \mathrm{dom}\,\varepsilon\}$ is an event in the sense of Definition 3.2. Note that condition

(e) If $h \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) \subseteq h'$, then $h' \in \mathrm{dom}\,\varepsilon$ and $\varepsilon(h) = \varepsilon(h')$

is equivalent to the conjunction of (b) and (d). Moreover, it is clear that all concepts defined in section 3 (cf., in particular, Definition 3.4) can be taken over to the revised concept without any problem.

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[8] Without going into detail, condition (b*) fits better to the approach to agency presented by Belnap et al. [3], since it allows for counting an agent's choices as events.

Following, events that fulfill condition (d) are said to be *success events*. When talking about success events we will often use the notations from section 3.

5. Mereological Properties

In this section we investigate some natural methods of constructing new events from others. For all these constructions the part relationship (with events as relata) is fundamental. However, when should an event ε count as a part of an event ε'? Obviously, any kind of part relation should define a (non-strict) partial order on the set of all events of the tree at hand, i. e., it should be reflexive, transitive, and anti-symmetrical.

We will focus on two part relations definable in terms of events:[9]

DEFINITION 5.1. Let ε and ε' be events in $\mathscr{T}$.

(a) ε is said to be a *phase* of ε' if $\mathrm{dom}\,\varepsilon' \subseteq \mathrm{dom}\,\varepsilon$ and if, for each $h \in \mathrm{dom}\,\varepsilon'$, $\varepsilon(h) \subseteq \varepsilon'(h)$.

(b) ε is said to be a *accompanied* by ε' if $\mathrm{dom}\,\varepsilon \subseteq \mathrm{dom}\,\varepsilon'$ and if, for each $h \in \mathrm{dom}\,\varepsilon$, $\varepsilon(h) \subseteq \varepsilon'(h)$. In this case we say that ε' is a *companion* of ε.

For example, the murder of Caesar is a companion of stabbing Caesar to death. And, heaving up the Peacock's anchor is a phase of the Peacock's leaving the harbor, which again is a phase of putting the Peacock to sea.

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[9] In terms of success events there are prima facie at least five possibilities for defining a part relation "E' is part of E", namely: (1) E' is a subset of E; (2) for each $e' \in E'$, there exists an $e \in E$ such that $e' \subseteq e$; (3) for each $e' \in E'$, there exists an $e \in E$ such that $e \subseteq e'$; (4) for each $e' \in E'$ and each history h with $e' \subseteq h$, there exists an $e \in E$ such that $e' \subseteq e \subseteq h$; and (5) $\mathrm{His}\langle E'\rangle$ is a subset of $\mathrm{His}\langle E\rangle$. Proposal (1) is obviously the strongest of them, since all the other conditions (2), (3), (4), and (5) are satisfied if E' is a subset of E. However, (1) excludes a priori too many examples where one is inclined to assume part relationship — for example, if each of both events is defined by one chain of moments and one of these chains contains the other as a proper subset. On the other hand, (5) is clearly too weak, since this condition does not even refer to the specific occurrences of an event, which depend on the histories in which the event occurs. But (5) can serve as a refutation condition: Any concept of part relation between events should guarantee that the set of histories in which the events at hand occur are comparable by the subset relation. From this point of view, proposal (2) can be withdrawn from the list above — consider, for example two events E and E', where E' is given by a single chain of moments, k', and E is defined by two disconnected chains k_1 and k_2 such that $k' \subseteq k_1$; furthermore, assume that there is a history that contains k' as subset, but does not include k_1. Then E' is part of E (in the sense of (2)), but the set of histories in which these events occur are not comparable by the subset relation.

As an immediate consequence of these different definitions of part relation we obtain the following two lemmata:

LEMMA 5.2. *The set of all events in $\mathscr{T}$ is (non-strictly) partially ordered by both relations that of phase and that of companion.* ■

LEMMA 5.3. *Let ε and ε' be events in $\mathscr{T}$ with* $\operatorname{dom}\varepsilon = \operatorname{dom}\varepsilon'$. *Then ε is a phase of ε' if and only if ε' is a companion of ε.* ■

PROPOSITION 5.4. *Let E and E' be success events in $\mathscr{T}$.*

(a) *E is a phase of E' if and only if for each $e' \in E'$, there exists an $e \in E$ with $e \subseteq e'$.*

(b) *E is a companion of E' if and only if for each chain $e' \in E'$ and each history h with $e' \subseteq h$, there exists a chain $e \in E$ such that $e' \subseteq e \subseteq h$.*

PROOF. (a) Let E be a phase of E', i.e., $\mathrm{His}\langle E'\rangle$ is a subset of $\mathrm{His}\langle E\rangle$ and $e_{E,h}$ is a subset of $e_{E',h}$ for each $h \in \mathrm{His}\langle E'\rangle$. Let now e' be arbitrarily chosen in E', and let h be a history that contains e' as subset. Then e' is identical with $e_{E',h}$, and hence there is an $e \in E$ such that $e \subseteq e'$, namely $e_{E,h}$.
For the other direction suppose that for each $e' \in E'$, there exists an $e \in E$ with $e \subseteq e'$. Obviously, $\mathrm{His}\langle E'\rangle$ is a subset of $\mathrm{His}\langle E\rangle$. Let now h be chosen in $\mathrm{His}\langle E'\rangle$. Consequently, there is an $e \in E$ such that $e \subseteq e_{E',h}$. Since e is a subset of h, h is in $\mathrm{His}\langle E\rangle$ and hence, by Lemma 3.3, $e_{E,h} = e \subseteq e_{E',h}$.
(b) Let E be a companion of E', i.e., $\mathrm{His}\langle E'\rangle$ is a subset of $\mathrm{His}\langle E\rangle$ and $e_{E',h}$ is a subset of $e_{E,h}$ for each $h \in \mathrm{His}\langle E'\rangle$. Consider a chain $e' \in E'$ and a history h with $e' \subseteq h$. Then h is contained in $\mathrm{His}\langle E\rangle$ and hence, by Lemma 3.3, $e' = e_{E',h} \subseteq e_{E,h} \subseteq h$.
For the other direction, assume that for each pair e', h with $e' \in E'$ and $e' \subseteq h$, there exists an $e \in E$ with $e' \subseteq e \subseteq h$. It is clear, then, that $\mathrm{His}\langle E'\rangle$ is a subset of $\mathrm{His}\langle E\rangle$. Consider now $h \in \mathrm{His}\langle E'\rangle$. Consequently, there exists an $e \in E$ with $e_{E',h} \subseteq e \subseteq h$. Then, by Lemma 3.3, e and $e_{E,h}$ are identical and hence $e_{E',h}$ is a subset of $e_{E,h}$. Thus, E is a companion of E'. ■

Following we will see that each event can be embedded into a convex event in a natural manner. Thereto, consider first a non-void set X that is partially ordered by a relation $<$. Then for each subset Y of X, there exists a least convex superset of Y, called *the convex hull* of Y. In more detail, the convex hull of Y is defined by:

$$Y^c := \bigcup_{\substack{y_1, y_2 \in Y, \\ y_1 \leq y_2}} \{\, y \in X \;:\; y_1 \leq y \leq y_2 \}.$$

Note that each intersection of convex sets and each union of a chain of convex sets is convex as well.

Theorem 5.5. *Each event ε in $\mathscr{T}$ has a least convex companion ε^c. In particular, ε^c is defined by:*

$$\operatorname{dom}\varepsilon^c := \operatorname{dom}\varepsilon \quad \text{and} \quad \varepsilon^c(h) := \varepsilon(h)^c.$$

Furthermore, if ε is finite (progressive, separated, bounded), then so ε^c.

Before proving this theorem, we will show the following simple lemma:

Lemma 5.6. *Let k be a chain in $\mathscr{T}$. Then for each history $h \in \text{His}$, k is a subset of h if and only if k^c is so.*

Proof. One direction holds trivially: Since k is a subset of k^c, each history that contains k^c also includes k. For the other direction, let m be chosen in k^c. Then there exist $m_1, m_2 \in k$ such that $m_1 \preccurlyeq m \preccurlyeq m_2$. Since m_2 is contained in h, m must be in h too. ∎

Proof of 5.5. First, we have to show that ε^c is an event.

Ad 4.1(a): By Lemma 5.6, we find that for each $h \in \operatorname{dom}\varepsilon^c$, $\emptyset \subsetneq \varepsilon(h) \subseteq \varepsilon^c(h) \subseteq h$.

Ad 4.1(b): Consider histories h and h' with $h \in \operatorname{dom}\varepsilon^c$ and $\varepsilon^c(h) \subseteq h'$. Consequently, $\varepsilon(h)$ is a subset of h'. Hence, by condition (b) applied to ε, h' is contained in $\operatorname{dom}\varepsilon$ and $\varepsilon(h')$ intersects with h. Obviously, $\varepsilon^c(h)$ also intersects with h.

Ad 4.1(c): Let $h, h' \in \operatorname{dom}\varepsilon^c$ such that $\varepsilon^c(h) \cap h'$ and $\varepsilon^c(h') \cap h$ are non-void. For symmetry reasons it is sufficient to show that $\varepsilon^c(h) \cap h'$ is a subset of $\varepsilon^c(h')$. By assumption, we obtain that $\varepsilon(h) \cap h'$ and $\varepsilon(h') \cap h$ are non-void and hence, by condition (d), that both intersections are identical. Without restriction we may assume that $\varepsilon^c(h)$ is not included in h'. Otherwise, $\varepsilon(h)$ would be a subset of h' and hence be a subset of $\varepsilon(h')$. Then $\varepsilon^c(h)$ and also $\varepsilon^c(h) \cap h'$ would be subsets of $\varepsilon^c(h')$ — and we are set. Since $\varepsilon^c(h)$ is not contained in h', $\varepsilon(h)$ is not included in h' either. Fix m^* in $\varepsilon(h) \setminus h'$. Let now m be arbitrarily chosen in $\varepsilon^c(h) \cap h'$. Then there are moments $m_1, m_2 \in \varepsilon(h)$ such that $m_1 \preccurlyeq m \preccurlyeq m_2$. From $m \in h \cap h'$ it follows that $m \prec m^*$. Since $\varepsilon(h) \cap h'$ and $\varepsilon(h') \cap h$ are identical, we obtain that $m_1 \in \varepsilon(h') \cap h$. Let us now assume for reductio ad absurdum that m were not in $\varepsilon^c(h')$. Then there would not exist any $m' \succcurlyeq m$ that is in $\varepsilon(h')$. Hence $\varepsilon(h') \prec m \in h \cap h'$ and consequently $\varepsilon(h) \prec m$ — contradicting our choice of m^*.

Thus, ε^c is an event that is convex and a companion of ε. We finally have to show that ε^c is the least convex companion of ε. For this, let ε' be an arbitrary convex companion of ε. Then ε' is also a companion of ε^c:

Obviously, $\operatorname{dom}\varepsilon^c$ is a subset of $\operatorname{dom}\varepsilon'$, since so is $\operatorname{dom}\varepsilon$. And, for each $h \in \operatorname{dom}\varepsilon^c$, $\varepsilon(h)$ is a subset of $\varepsilon'(h)$. Since $\varepsilon^c(h)$ is the smallest convex superset of $\varepsilon(h)$, it follows that $\varepsilon^c(h)$ is a subset of $\varepsilon'(h)$. ■

Now we turn to discuss conditions by which an event can be embedded into a finite event. It is prima facie clear that this project can be successful only if the considered event is bounded (cf. Definition 3.4): An event ε is bounded if and only if for each history $h \in \operatorname{dom}\varepsilon$ there are moments $m, m' \in h$ such that $m \preccurlyeq \varepsilon(h) \preccurlyeq m'$.

THEOREM 5.7. *Let $\mathscr{T}$ be a sup-closed tree. Then each bounded and separated event ε has a least finite companion ε^f defined by:*

$$\begin{aligned} \operatorname{dom}\varepsilon^f &:= \operatorname{dom}\varepsilon \\ \varepsilon^f(h) &:= \varepsilon(h) \cup \{\inf\varepsilon(h), \sup_h \varepsilon(h)\}. \end{aligned}$$

Furthermore, if ε is convex, then so is ε^f.

PROOF. First, ε^f obviously satisfies condition 4.1(a) and (b). For proving condition 4.1(c), let m and m' be moments with $m \in \varepsilon^f(h) \cap h'$ and $m' \in \varepsilon^f(h') \cap h$. Then $\varepsilon(h) \cap h'$ is non-void if and only if so is $\varepsilon(h') \cap h$. For if, for example, $\varepsilon(h) \cap h'$ is void and $\varepsilon(h') \cap h$ is not, then $m = \inf\varepsilon(h)$ and hence $\emptyset \neq \varepsilon(h') \cap h \prec \varepsilon(h)$. Since ε is separated, there exists a moment $m'' \in h \setminus h'$ with $\varepsilon(h') \cap h \prec m'' \preccurlyeq \varepsilon(h)$. Hence $m'' \preccurlyeq \inf\varepsilon(h)$ and consequently $\inf\varepsilon(h) \notin h'$—in contradiction to the choice of m. Hence we only have to consider the following cases:

Case 1: $\varepsilon(h) \cap h' = \emptyset$ and $\varepsilon(h') \cap h = \emptyset$. From these assumption we obtain that $h \cap h' \prec \varepsilon(h), \varepsilon(h')$. Hence $\inf\varepsilon(h) = m = m' = \inf\varepsilon(h')$ and thus $\varepsilon^f(h) \cap h' = \{\inf\varepsilon(h)\} = \{\inf\varepsilon(h')\} = \varepsilon^f(h') \cap h$.

Case 2: $\varepsilon(h) \cap h' \neq \emptyset$ and $\varepsilon(h') \cap h \neq \emptyset$. In this case we get that $\varepsilon(h) \cap h' = \varepsilon(h') \cap h$. Then $\inf\varepsilon(h) = \inf(\varepsilon(h) \cap h') = \inf(\varepsilon(h') \cap h) = \inf\varepsilon(h') \in \varepsilon^f(h') \cap h$. If $\sup_h \varepsilon(h)$ is not in $\varepsilon(h) \cap h'$, we obtain that $\varepsilon(h) = \varepsilon(h')$ (by condition 4.1(c) applied to ε) and hence that $\sup_h \varepsilon(h) = \sup_{h'} \varepsilon(h')$, i.e., $\sup_h \varepsilon(h)$ is contained in $\varepsilon^f(h') \cap h$.

In both cases $\varepsilon^f(h) \cap h'$ is a subset of $\varepsilon^f(h')$. For symmetry reasons it follows that $\varepsilon^f(h) \cap h'$ and $\varepsilon^f(h') \cap h$ are identical.

All other claims hold trivially. ■

PROPOSITION 5.8. *Let $\mathscr{T}$ be a sup-closed dense tree. Then $\mathscr{T}$ is of splitting type 2 if and only if each event in $\mathscr{T}$ is separated.*

PROOF. Let us first assume that each event in $\mathscr{T}$ is separated. Let h and h' be distinct histories that intersect. Consider then the event ε defined by

$\varepsilon(h) := h \setminus h'$ and $\varepsilon(h') := h'$. Hence, $\emptyset \neq h \cap h' = \varepsilon(h') \cap h \prec \varepsilon(h)$. Since this event is separated, there exists an $m \in h \setminus h'$ such that $\varepsilon(h') \cap h \prec m \preccurlyeq \varepsilon(h)$. Since $h \cap h'$ is upper-bounded, the supremum of this set in h exists, and we get that $\sup_h h \cap h' = \inf h \setminus h' = m \notin h'$ (note that $\mathscr{T}$ is dense). Hence, $h \cap h'$ does not have a maximum.

For the other direction, let $\mathscr{T}$ be of splitting type 2 and consider an event ε in $\mathscr{T}$. We have to show that ε is separated. Thereto, let h and h' be histories in $\operatorname{dom} \varepsilon$ such that $\emptyset \neq \varepsilon(h') \cap h \prec \varepsilon(h)$. Then $\inf \varepsilon(h)$ exists, and $\varepsilon(h') \cap h \preccurlyeq \inf \varepsilon(h) \preccurlyeq \varepsilon(h)$. Note that $\varepsilon(h)$ does not intersect with h'. Otherwise, $\varepsilon(h) \cap h'$ and $\varepsilon(h') \cap h$ would be non-void and hence they would be identical. But this would entail that $\varepsilon(h) \cap h' \prec \varepsilon(h)$ — contradicting that these sets are non-void. From $\varepsilon(h) \cap h' = \emptyset$ it follows that $h \cap h' \prec \varepsilon(h)$. Now, if $\inf \varepsilon(h)$ were in h', $\inf \varepsilon(h)$ would be the maximum of $h \cap h'$. But this contradicts that $\mathscr{T}$ is of splitting type 2. Thus, $\inf \varepsilon(h)$ is contained in $h \setminus h'$ and consequently $\varepsilon(h') \cap h \prec \inf \varepsilon(h) \preccurlyeq \varepsilon(h)$. ■

In this section, we have discussed two constructions that seem natural for the approach to event presented in section 4. Note that the universe of all events in a given tree is not closed with respect to boolean operations, i. e., there is no general way of defining the intersection or the union of events. Mereological (partial) functions such as the sum and the product could be defined as follows: The *sum* of events ε and ε' is the least event that has both events ε and ε' as phases. The *product* of events ε and ε' is the greatest common phase of both events. In general, however, a pair of events need not to have a common phase. To see this, consider the success events E and E' depicted in figure 3. Which event definable in that tree could then count as

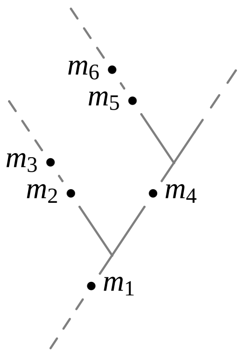

Figure 3. Common phases. Let $E := \{\{m_1, m_2\}, \{m_4, m_5\}\}$ and $E' := \{\{m_1, m_3\}, \{m_4, m_6\}\}$. Which events are phases of the events E and E'?

a common phase of both events?

6. Events as Propositions

In this section we investigate how events correspond to propositions. We start with a precise definition of the concept of proposition:

DEFINITION 6.1 (*Proposition*). Let $\mathscr{T}$ be a tree. A *proposition* in $\mathscr{T}$ is a subset of $\{ (m,h) \in \text{Mom} \times \text{His} \ : \ m \in h\}$. A proposition P in $\mathscr{T}$ is said to be *eternal* if for each history h and all moments $m, m' \in h$, it holds that $(m,h) \in P$ if and only if $(m',h) \in P$.

The basic idea here is that a proposition P can be represented by the set of all moment-history pairs (m,h), where P holds true at moment m in history h. Obviously, eternal propositions are propositions that fulfill the following condition: P holds true at each moment m in history h if it holds true at at least one moment in that history. Hence, eternal propositions may also be represented by sets of histories.

In terms of synchronized trees, it seems more natural to represent propositions by sets of instant-history pairs (instead of sets of moment-history pairs). But both approaches are equivalent since there exists a one-to-one correspondence between the sets $\text{Inst} \times \text{His}$ and $\{ (m,h) \in \text{Mom} \times \text{His} \ : \ m \in h\}$, which is given by the assignments $(i,h) \mapsto (m_{i,h}, h)$ and its inverse $(m,h) \mapsto (i_m, h)$.

Let now ε be an event in $\mathscr{T}$. Obviously,

$$P_\varepsilon = \{ (m,h) \ : \ h \in \text{dom}\, \varepsilon \text{ and } m \in \varepsilon(h)\}$$

defines a proposition in $\mathscr{T}$, namely the proposition expressed by "ε currently runs". It is easy to show that the assignment $\varepsilon \mapsto P_\varepsilon$ is injective. Moreover, if ε is accompanied by ε', then P_ε is a subset of $P_{\varepsilon'}$.

DEFINITION 6.2. A non-void proposition P in $\mathscr{T}$ is said to be *event-like* if there exists an event ε such that $P = P_\varepsilon$.

By this definition, it is clear that each event defines an event-like proposition. But what are the criteria that make a proposition event-like? To investigate this, we first note that the event $P := P_\varepsilon$ satisfies the following conditions:

(P1) Let h and h' be histories such that h' contains each moment m' with $(m',h) \in P$. Then if (m,h) is contained in P, so is (m,h').

(P2) Let m and m' be moments in $h \cap h'$. If both pairs (m,h) and (m',h') are contained in P, then so is (m,h') (as well as (m',h)).

Vice versa, assume that a non-void proposition P meets these conditions. Then proposition P defines an event ε_P by:

$$\operatorname{dom} \varepsilon_P := \{ h \in \text{His} : \text{there is an } m \in h \text{ with } (m, h) \in P \},$$
$$\varepsilon_P(h) := \{ m \in h : (m, h) \in P \}.$$

Note that the assignments $\varepsilon \mapsto P_\varepsilon$ and $P \mapsto \varepsilon_P$ are inverse to each other. Thus, we obtain that a proposition P is event-like if and only if ε_P is an event. Moreover, the assignment $P \mapsto \varepsilon_P$ preserves part relationship: If P and P' are propositions such that $P \subseteq P'$ and both P and P' fulfill conditions (P1) and (P2), then event ε_P is accompanied by event $\varepsilon_{P'}$.

In summary, we obtain the following theorem:

Theorem 6.3. *A non-void proposition P is event-like if and only if P satisfies conditions (P1) and (P2).*

If the tree under consideration is synchronized and if propositions are represented by sets of instant-history pairs, the concept of event-like propositions can be reformulated in the following way: Let $P \subseteq \text{Inst} \times \text{His}$ be a proposition. For histories h and h', let $\tau_{P,h}$ be the set of all instants i with $(i, h) \in P$ and let $\tau_{h,h'}$ be the set of those instants i such that $m_{i,h}$ is identical with $m_{i,h'}$. Then proposition P is event-like if and only if P is a non-void proposition satisfying:

(P1') If $\tau_{P,h}$ is contained in $\tau_{h,h'}$, then it is also contained in $\tau_{P,h'}$.

(P2') If both instant sets $\tau_{P,h} \cap \tau_{h,h'}$ and $\tau_{P,h'} \cap \tau_{h,h'}$ are non-void, then $\tau_{P,h} \cap \tau_{h,h'} \subseteq \tau_{P,h'}$.

Since the proof of equivalence of these concepts is straightforward, it can be omitted here.

Let us briefly summarize the results of this and the previous section: Though events have propositional correlata, events do not form a subcategory of the category of propositions, since they do not share typical properties of propositions — particularly with regard to fact that the class of event-like propositions is not closed with respect to boolean operations.

7. Xu Events

Xu [32] presents a concept of event that is based on a theory of *transition*. To sketch this approach, let us consider a tree $\mathscr{T}$. An *initial* in $\mathscr{T}$ is a non-void chain of moments, p, that is upper-bounded and backward closed, i. e., there exists a moment m such that $p \preccurlyeq m$, and m' is an element of p provided there is an $m \in p$ with $m' \prec m$. An *outcome* is a non-void forward closed and connected set of moments, F, i. e.:

(a) For each $m \in F$ and each $m' \succ m$, $m' \in F$.

(b) For all moments $m, m' \in F$, there is an $m'' \in F$ with $m'' \preccurlyeq m, m'$.

Then a *transition* is an ordered pair $\langle p, F\rangle$, where p is an initial, F an outcome, and $p \prec F$. A transition $\langle p, F\rangle$ is said to *occur* in history h if F intersects with h (note that in this case p is a subset of h).

Definition 7.1 (cf. Xu [32]). An *X-event* in $\mathscr{T}$ is a non-void set χ of transitions in $\mathscr{T}$ such that for each history $h \in \text{His}$, there exists at most one transition $\langle p, F\rangle \in \chi$ that occurs in h. $\text{His}\langle\chi\rangle$ is defined as the set of all histories h such that there is a transition in χ that occurs in h.

This concept of event differs from the ones presented previously since there is no general way of translating events in the sense of Definition 3.2 or 4.1 respectively into X-events, or vice versa. However, translations can be found if we restrict consideration to the following situation: Let $\mathscr{T}$ be a sup-closed and dense tree of splitting type 2 or 4. Let χ be an X-event. Then for each history $h \in \text{His}\langle\chi\rangle$, there exists exactly one transition $\tau_h = \langle p_h, F_h\rangle$ such that F_h intersects with h. Note that for each history $h \in \text{His}\langle\chi\rangle$, the transition phase $\delta_{\tau_h} := \{\, m : p_h \prec m \preccurlyeq F_h\}$ is non-void. In what follows we assume that χ also meets the following condition:

(∗) If $\langle p, F\rangle, \langle p, F'\rangle \in \chi$ are transitions with $p \subseteq p'$, then p and p' are identical.

Then χ defines an event by:

$$\text{dom}\, \varepsilon_\chi := \text{His}\langle\chi\rangle \quad \text{and} \quad \varepsilon_\chi(h) := \delta_{\tau_h}.$$

The crucial point is to verify that condition 4.1 (b) is satisfied. To prove this, let $\varepsilon_\chi(h) = \delta_{\tau_h}$ be a subset of history h'. We have to show that $h' \in \text{dom}\, \varepsilon_\chi$ and that $\varepsilon_\chi(h')$ intersects with h. It suffices to show that $F_h \cap h$ intersects with h' because in this case the claim holds trivially. For reductio ad absurdum let us assume that $F_h \cap h$ is contained in $h \setminus h'$. Since δ_{τ_h} is a subset of h', $F_h \cap h$ and $h \setminus h'$ must be identical. As the underlying tree has splitting type 4, we obtain that $\inf(F_h \cap h) = \inf(h \setminus h') \in h \setminus h' = F_h \cap h$. But obviously, $\inf(F_h \cap h) \in \delta_{\tau_h} \subseteq h'$. Hence $\inf(F_h \cap h)$ must be contained in h' and in $h \setminus h'$ — a contradiction.

Note that ε_χ is convex. Moreover, $\text{His}\langle\varepsilon_\chi\rangle$ and $\text{His}\langle\chi\rangle$ are identical. For if h is a history and $\tau = \langle p, F\rangle \in \chi$ is a transition with $F \cap h \neq \emptyset$, then δ_τ is a subset of h, and hence $h \in \text{His}\langle\varepsilon_\chi\rangle$. And, if δ_τ is subset of a history

h', then $\sup_{h'} \delta_\tau = \inf(F \cap h')$ is the minimum of $F \cap h'$, and consequently $h' \in \mathrm{His}\langle\chi\rangle$.

Vice versa, let us now consider a convex event ε. Then for each $h \in \mathrm{dom}\,\varepsilon$, the ordered pair $\tau_h := \langle p_h, F_h\rangle$ is a transition, where p_h and F_h are defined by:

$$p_h := \{\, m : m \prec \varepsilon(h)\} \quad \text{and} \quad F_h := \{\, m : \varepsilon(h) \preccurlyeq m\}.$$

By condition 4.1 (c), for all h, h' in $\mathrm{dom}\,\varepsilon$ with $p_h \subseteq p_{h'}$, p_h and $p_{h'}$ must be identical. Hence, we can define an X-event

$$\chi_\varepsilon := \left\{\, \tau_h : h \in \mathrm{dom}\,\varepsilon \,\right\}$$

that satisfies $\mathrm{His}\langle\varepsilon\rangle = \mathrm{His}\langle\chi_\varepsilon\rangle$ and condition $(*)$.

It is worthwhile to note that the translations just presented are not unique. A quite different one is the following: Let χ be an X-event that satisfies condition $(*)$. Then for each history h and all pairs $\langle p, F\rangle, \langle p', F'\rangle \in \chi$ with $F \cap h \neq \emptyset$ and $F' \cap h \neq \emptyset$, it holds that $p = p'$ and $F = F'$. Define then

$$\mathrm{dom}\,\varepsilon_\chi := \mathrm{His}\langle\chi\rangle \quad \text{and} \quad \varepsilon_\chi(h) := p_h \cup (F_h \cap h),$$

where $\langle p_h, F_h\rangle$ denotes the unique transition in χ with $F_h \cap h \neq \emptyset$. It is easy to prove that ε_χ is an event in the sense of Definition 4.1: Here conditions (a) and (b) hold trivially. For condition (c) assume that both sets $\varepsilon_\chi(h) \cap h'$ and $\varepsilon_\chi(h') \cap h$ are non-void. Without restriction we may assume that $p_h = p_{h'}$. Otherwise, $h \cap h'$ is a proper subset of both p_h and $p_{h'}$. Then it holds that $p_h \cap h' = p_{h'} \cap h$, $F_h \cap h' = \emptyset$, and $F_{h'} \cap h = \emptyset$, i. e., (c) is satisfied. We may also assume that both $F_h \cap h'$ and $F_{h'} \cap h$ are void. For if, for example, $F_h \cap h'$ is non-void, the transition $\langle p_h, F_h\rangle$ occurs in h' and hence is identical with $\langle p_{h'}, F_{h'}\rangle$ — and consequently condition (c) holds trivially. From this we can conclude that $\varepsilon_\chi(h) \cap h' = p_h \cap h' \subseteq p_{h'} \subseteq \varepsilon_\chi(h')$.

8. States

The concept of tree as considered up to now entails only a rudimentary state concept. The best of way of representing states is to define them as arbitrary sets of moments. But this admits that each chain and, in particular, each history count as a state. In order to avoid this, we introduce a new notion of tree:

DEFINITION 8.1. A *state tree* is an ordered triple $\mathscr{T} = \langle \mathrm{Mom}, \prec, \mathrm{Tot}\rangle$, where $\langle \mathrm{Mom}, \prec\rangle$ is a tree and Tot is a partition of Mom satisfying the following condition:

(S0) Let k be a non-void and upper-bounded chain in Mom, and let h and h' be histories containing k. If both $\sup_h k$ and $\sup_{h'} k$ exist and if there is an $s \in \text{Tot}$ with $\sup_h k\,, \sup_{h'} k \in s$, then $\sup_h k = \sup_{h'} k\,.$

The elements of Tot are said to be *total states.* Moments m and m' are *state equivalent* if there is an $s \in \text{Tot}$ that contains both m and m'.

Condition (S0) requires that, if h and h' coincide along k and if h and h' are separated in the "next" moments after k (in h and h' respectively), then h and h' must have a different total state in these next moments. To put it another way: there must be a reason why histories h and h' are separated after k, and this reason is somehow encoded in the total states that are realized immediately after k. Note that, for example, each tree with splitting type 1, trivially satisfies this condition.

States are then introduced as follows:

DEFINITION 8.2 (*States).* Let $\mathscr{T}$ be a state tree. A set of moments, s, is said to be a *state* in $\mathscr{T}$ if it closed with respect to state equivalence, i. e., if for each $s' \in \text{Tot}$ with $s \cap s' \neq \emptyset$, s' is a subset of s. The set of all states is denoted by Stat. A state s is said to be *realized* at moment m if m is an element of s. A state s is a *sub-state* of state s' if s' is a subset of s, i. e., if s is realized whenever s' is realized.

Hence, each state is a (possibly void) union of total states, and vice versa. In particular, each total state is a state and so are $\emptyset$ and Mom. Furthermore, it is clear that the set of all states forms a complete boolean algebra.

DEFINITION 8.3. A *synchronized state tree* is a synchronized tree $\mathscr{T} = \langle \text{Mom}, \prec, \text{Inst} \rangle$ on which a partition Tot is defined that satisfies (S0).

We do not impose further requirements on the relationship between states and instants. For example, the condition

(S1) Each instant is a state

seems too strong, since it entails that in each history total states can never be realized twice:

PROPOSITION 8.4. *Let $\mathscr{T}$ be a synchronized state tree satisfying (S1). Then there are no state-equivalent moments $m, m' \in \text{Mom}$ with $m \prec m'$.*

PROOF. Assume for reductio ad absurdum that there were state-equivalent moments m and m' with $m \prec m'$, i. e., there is an $s \in \text{Tot}$ with $m, m' \in s$. By (S1), instant i_m is a state in $\mathscr{T}$. From $m \in s \cap i_m$ it then follows that s is a subset of i_m. Consequently, $m' \in i_m$ and hence $i_m = i_{m'}$. But, since there is a history that contains both m and m', m and m' must be identical — contradicting the irreflexivity of $\prec$. ■

Proposition 8.4 also holds true if we replace (S1) by condition:

(S2) For each total state s there is an instant i with $s \subseteq i$.

This would mean that total states carry explicit information about its temporal realizability.

Definition 8.5. A synchronized state tree is said to be *state-history complete* if, for each map π : Inst $\rightarrow$ Tot, there exists a history h_π with $m_{i,h_\pi} \in \pi(i)$ for each $i \in$ Inst.

Proposition 8.6. *A synchronized state tree is state-history complete if and only if for each map* π : Inst $\rightarrow$ Stat $\setminus\{\emptyset\}$, *there is a history* h_π *such that* $m_{i,h_\pi} \in \pi(i)$ *for each* $i \in$ Inst.

Proof. Let π : Inst $\rightarrow$ Stat $\setminus\{\emptyset\}$ be a map. As said before, each state is a union of total states. By the Axiom of Choice, we then obtain a map $\hat{\pi}$: Inst $\rightarrow$ Tot with $\hat{\pi}(i) \subseteq \pi(i)$ for each instant i. Hence, by assumption, there exists a history $h_{\hat{\pi}}$ with $m_{i,h_{\hat{\pi}}} \in \hat{\pi}(i)$ for each $i \in$ Inst. Thus, $m_{i,h_{\hat{\pi}}} \in \pi(i)$ for each $i \in$ Inst.

The other direction holds trivially. ■

9. Meixner Events

The last approach to events discussed here is one proposed by Meixner [16]. According to his approach, events are defined via state sequences. We can reconstruct this idea here as follows:

Definition 9.1. Let $\mathscr{T}$ be a synchronized state tree. An *M-event* in $\mathscr{T}$ is a partial function μ with non-void domain dom $\mu \subseteq$ Inst that assigns to each $i \in$ dom μ a non-void state $\mu(i) \in$ Stat.

An M-event μ *occurs* in history $h \in$ His if for each $i \in$ dom μ, $\mu(i)$ is realized at moment $m_{i,h}$. Let His$\langle\mu\rangle$ denote the set of all histories in which μ occurs.

As an immediate consequence we obtain the following characterization:

Lemma 9.2. *Let* $\mathscr{T}$ *be a synchronized state tree. Then the following statements are equivalent:*

(i) $\mathscr{T}$ *is state-history complete.*

(ii) *Each M-event in* $\mathscr{T}$ *occurs in at least one history.*

PROOF. Suppose first that $\mathscr{T}$ is state-history complete and that μ is an M-event. Define a map $\pi : \mathrm{Inst} \to \mathrm{Stat} \setminus \{\emptyset\}$ by $\pi|_{\mathrm{dom}\,\mu} = \mu$ and $\pi(i) := s_0$ for each instant $i \in \mathrm{Inst} \setminus \mathrm{dom}\,\mu$, where s_0 is an arbitrarily fixed non-void state. By Proposition 8.6, there exists a history h_π with $m_{i,h_\pi} \in \pi(i)$ for each $i \in \mathrm{Inst}$. Obviously, μ occurs in h_π.

The other direction holds trivially, since each map $\pi : \mathrm{Inst} \to \mathrm{Stat} \setminus \{\emptyset\}$ is an M-event. ■

DEFINITION 9.3. Let μ and μ be M-events in a synchronized state tree $\mathscr{T}$. μ is said to be a *sub-event* of μ' if $\mathrm{dom}\,\mu \subseteq \mathrm{dom}\,\mu'$ and $\mu'(i) \subseteq \mu(i)$ for each $i \in \mathrm{dom}\,\mu$.

Obviously, if μ is a sub-event of μ', $\mathrm{His}\langle\mu'\rangle$ is contained in $\mathrm{His}\langle\mu\rangle$. For if μ' occurs in history h, we can conclude that $m_{i,h} \in \mu'(i)$ for each $i \in \mathrm{dom}\,\mu'$. Hence $m_{i,h} \in \mu'(i)$ for each $i \in \mathrm{dom}\,\mu$, and consequently $m_{i,h} \in \mu(i)$ for each $i \in I$. Thus, μ occurs in h.

PROPOSITION 9.4. *Let $\mathscr{T}$ be a synchronized state tree and let μ be an M-event with* $\mathrm{His}\langle\mu\rangle \neq \emptyset$. *Then*

$$E_\mu := \{\, e_{\mu,h} \ :\ h \in \mathrm{His}\langle\mu\rangle \,\}, \quad \textit{with } e_{\mu,h} := \hat{h}(\mathrm{dom}\,\mu),$$

is a success event in $\mathscr{T}$ (cf. Definition 3.2). Moreover, it holds:

(a) $\mathrm{His}\langle\mu\rangle$ *is identical with* $\mathrm{His}\langle E_\mu\rangle$.

(b) *If μ is a sub-event of an M-event μ', then E_μ is a phase of $E_{\mu'}$.*

PROOF. Obviously, for each $h \in \mathrm{His}\langle\mu\rangle$, $e_{\mu,h}$ is a non-void chain of moments. Hence, E_μ is a non-void set of non-void chains in Mom. Furthermore, it holds that $e_{\mu,h} = e_{\mu,h'}$ if $e_{\mu,h} \subseteq h'$. Assume now that μ occurs in histories h and h' and that both $e_{\mu,h} \cap h'$ and $e_{\mu,h'} \cap h$ are non-void. For symmetry reasons it is sufficient to show that $e_{\mu,h} \cap h'$ is a subset of $e_{\mu,h'}$. Let m be arbitrarily chosen in $e_{\mu,h} \cap h'$. Then $m = m_{i,h}$ for some $i \in \mathrm{dom}\,\mu$ and hence $m_{i,h} = m_{i,h'}$. Since μ occurs in h', $m = m_{i,h'}$ must be in $e_{\mu,h'}$ too.

(a) Let h be arbitrarily chosen in $\mathrm{His}\langle\mu\rangle$. Then $e_{\mu,h} \in E_\mu$ and hence, by $e_{\mu,h} \subseteq h$, $h \in \mathrm{His}\langle E_\mu\rangle$. Vice versa, let h be a history in which E_μ occurs. Then there exists an $e \in E_\mu$ with $e \subseteq h$. From the definition of E_μ we obtain that there exists an $h' \in \mathrm{His}\langle\mu\rangle$ such that $e = e_{\mu,h'}$. Consequently, $e_{\mu,h'} \subseteq h$ and hence $e_{\mu,h'} = e_{\mu,h}$. Hence for each $i \in \mathrm{dom}\,\mu$, $m_{i,h} = m_{i,h'} \in \mu(i)$ and thus μ occurs in h.

(b) If μ is sub-event of μ', then $\mathrm{His}\langle\mu'\rangle$ is contained in $\mathrm{His}\langle\mu\rangle$ and hence, by (a), $\mathrm{His}\big\langle E_{\mu'}\big\rangle$ is a subset of $\mathrm{His}\langle E_\mu\rangle$. And for each $h \in \mathrm{His}\big\langle E_{\mu'}\big\rangle$, $e_{\mu,h}$ is a subset of $e_{\mu',h}$, since $\mathrm{dom}\,\mu$ is contained in $\mathrm{dom}\,\mu'$. ■

10. Summary

In this paper we studied a new concept of event and discussed its formal properties. We started by generalizing step-by-step a definition of event proposed by Kutschera [10]. The reason for doing so was that Kutschera's event concept is apparently too restrictive to subsume important classes of happenings. We proved that Kutschera events form a proper subclass of the class of events in the sense proposed here. Moreover, we showed that many events can be represented by Kutschera events.

In a second step we compared other event conceptions to our notion of event. In particular, we saw that Meixner events can be represented as events if the notion of tree is enriched by a state concept. On the other hand, we found that there is only a loose relationship between Xu events and events in our sense.

This paper focused on the analysis of singular events. A detailed analysis of generic events, i. e., events that may occur several times (such as Tim's opening the window), would be worth pursuing. There seem to be at least two strategies for defining such events: First one may represent generic events by sets of singular events. Second we may define generic events by relaxing the conditions for singular events, i. e., by assigning a set of distinct occurrences to each history in which the event happens. It would be worth investigating whether both approaches result in equivalent concepts.

Many event descriptions include spatial information, which leads some authors to think that events necessarily have spatial or spatio-temporal parts. In that case events could be analyzed in a satisfactory manner only in an ontological background theory that allows for talking about spatial entities (cf., e. g., Lewis [11]). We did not adopt this view, but an analysis of *spatial events* would obviously be an interesting enterprise. For such an analysis, Belnap's *branching space-time theory* (cf. Belnap [2]) would provide a rich background theory.

Another field for further research is an analysis of causal notions based on the event approach presented here. It is worthwhile to note that an analysis of the causal relation in terms of Meixner and Kutschera events has been carried out by Meixner [17]. The feature of that approach is that general causal concepts can be distinguished by employing the causal structure that is already built in the structure of branching-time models.

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