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COMBINATIONS OF TENSE AND MODALITY FOR PREDICATE LOGIC

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ABSTRACT. In recent years combinations of tense and modality have moved into the focus of logical research. From a philosophical point of view, logical systems combining tense and modality are of interest because these logics have a wide field of application in original philosophical issues, for example in the theory of causation, of action, etc. But until now only methods yielding completeness results for propositional languages have been developed. In view of philosophical applications, analogous results with respect to languages of predicate logic are desirable, and in this paper I present two such results. The main developments in this area can be split into two directions, differing in the question whether the ordering of time is world-independent or not. Semantically, this difference appears in the discussion whether T×W-frames or Kamp-frames (resp. Ockham-frames) provide a suitable semantics for combinations of tense and modality. Here, two calculi are presented, the first adequate with respect to Kamp-semantics, the second to T×W-semantics. (Both calculi contain an appropriate version of Gabbay's irreflexivity rule.) Furthermore, the proposed constructions of canonical frames simplify some of those which have hitherto been discussed.

KEY WORDS: tense, modality, Kamp-semantics, T×W-semantics, (strong) completeness

INTRODUCTION

In recent years combinations of tense and modality have moved into the focus of logical research. From a philosophical point of view, logical systems combining tense and modality (short: "temporal modal logics") are of interest because these logics have a wide field of application in original philosophical issues, for example in the theory of causation, of action, etc. But until now only methods yielding completeness results for propositional languages have been developed. In view of philosophical applications, analogous results with respect to languages of predicate logic are desirable, and in this paper I present two such results.

The main developments in this area can be split into two directions, differing in the question whether the ordering of time is world-independent or not. Semantically, this difference appears in the discussion whether T×W-frames or Kamp-frames (resp. Ockham-frames) provide a suitable semantics for combinations of tense and modality. Here, two calculi are

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presented, the first adequate with respect to Kamp-semantics, the second to T×W-semantics. (Both calculi contain an appropriate version of Gabbay's irreflexivity rule.) Furthermore, the proposed constructions of canonical frames simplify some of those which have hitherto been discussed.

In Kamp-semantics each world has its own order of time, and these orders can coincide, but need not. In T×W-semantics "coincidence of orders of time" has to be expressed by introduction of an additional logical symbol $\Box$. Syntactically, this symbol is characterized as a unary operator, analogous to the symbols G (*it will always be that...*), H (*it has always been that...*) and N (*it is historically necessary that...*). Semantically, there are two possibilities to read $\Box\varphi$:

(a) φ is true in every world – this reading (and a corresponding calculus) has been first discussed by Di Maio and Zanardo (1997);
(b) φ is true in the actual world – see Kutschera (1997).

For both readings an axiomatization and a completeness proof – with respect to propositional languages – are given in Kutschera (1997). In his completeness proofs Kutschera makes essential use of Gabbay's irreflexivity lemma.[1] I am not sure whether this lemma could be applied to languages of predicate logic as well. However, we will see that no further problem arises in extending the results of Kutschera to languages of predicate logic. In what follows, we will restrict ourselves to the first reading of $\Box\varphi$.

In quantified modal logic, there is a broad discussion about an intuitively satisfactory semantics for modal logic, and correspondingly, of the formulas which are justified by the semantics in question. It is not the aim of this paper to unfold these problems with respect to temporal modal logic, although all the problems of quantified modal logic reappear in quantified temporal modal logic. Thus, we restrict the semantics as follows:

(a) We read the quantifier in a "possibilist" way, $\forall x$ in the sense of "for all possible objects $x\ldots$" and $\exists x$ as "there is a possible object $x\ldots$". Thus in every world and every moment, we have the same universe of discourse of individuals (= possible objects) over which the quantifiers range. Hence, "quantifying in" and the Barcan-formula are unproblematic. The usual "actualist" quantification can then be expressed in the following way: introduce an additional existence predicate E and relativize the global possibilist quantification to that predicate, i.e. define $\forall^a x\varphi := \forall x(Ex \rightarrow \varphi)$ and $\exists^a x\varphi := \exists x(Ex \wedge \varphi)$.[2]
(b) We interpret individual constants as proper names and therefore as rigid designators, i.e. every constant denotes the same individual in every world at every moment, even though this object may be nonex-

isting in some world at some moment; for example: “John F. Kennedy” denotes now in the actual world a nonexisting object. The introduction of nonrigid designators (e.g. descriptions) does not present serious problems. There are techniques to eliminate nonrigid designators in quantified modal logic, and these techniques can be used with respect to temporal modal logic as well.[3]

1. PRELIMINARIES

In this paper we will establish completeness results parallely for Kamp- and T×W-semantics. Therefore we will consider two languages $\mathcal{L}$ and $\mathcal{L}_\Box$. The alphabet of the first one consists of:

(a) countably many (individual) constants c_n,
(b) for every natural number $m > 0$, countably many m-ary relation symbols R^m_n, but denumerably many unary relation symbols,
(c) denumerably many (individual) variables x_n,
(d) the symbols: E $\doteq$,
(e) the operators: $\neg$ $\rightarrow$ $\forall$ G H N,
(f) the auxiliary symbols:) (,.

The alphabet of $\mathcal{L}_\Box$ contains as additional symbol the operator $\Box$. Variables and constants are called **terms**. **Formulas** – i.e. $\mathcal{L}$**-formulas** resp. $\mathcal{L}_\Box$**-formulas** – are precisely those strings of finite length which can be obtained by the following rules:

(a) If s and t are terms, then Es and $s \doteq t$ are formulas.
(b) If R is a m-ary relation symbol and $t_1, \dots, t_m$ are terms, then $R(t_1, \dots, t_m)$ is a formula.
(c) If φ and ψ are formulas, then so are $\neg\varphi$ and $(\varphi \rightarrow \psi)$.
(d) If φ is a formula, then Gφ, Hφ and Nφ are formulas, and $\Box\varphi$ is a $\mathcal{L}_\Box$-formula.
(e) If φ is a formula and x is a variable, then $\forall x\varphi$ is a formula.

We use the symbols $\dot{\neq}$, $\wedge$, $\vee$, $\leftrightarrow$, $\exists$ and $\bot$ as (metalinguistic) abbreviations defined in the usual manner. Furthermore, we set F := $\neg$G$\neg$, P := $\neg$H$\neg$, M := $\neg$N$\neg$ and $\Diamond := \neg\Box\neg$. As noted in the introduction, we define:

$$\forall^a x\varphi := \forall x(Ex \rightarrow \varphi),$$
$$\exists^a x\varphi := \exists x(Ex \wedge \varphi).$$

The concepts of **free variable** and **sentence** are understood as usual. Given a formula φ, a variable x and a term t, let $\varphi(x/t)$ be that formula which is obtained by replacing every free occurrence of x in φ by t. Analogously, we understand $\varphi(c/x)$ to be that formula which results by replacing each occurrence of the constant c in φ by the variable x. A variable y is said to be **free for** the variable x in a formula φ iff no free occurrence of x lies within the "scope" of a quantifier $\forall y$ that has been used to build φ according to the rules above. Furthermore, let every constant c be **free for** x in φ.

2. THE CALCULI $\boldsymbol{K}^{\mathrm{irr}}$ AND $\boldsymbol{K}^{\mathrm{irr}}_{\Box}$

The **calculus** $\boldsymbol{K}$ combines the usual calculus of predicate logic with the propositional calculus of historical necessity. It includes the following axioms and rules (more exactly schemata of axioms and rules):

Axioms of $\boldsymbol{K}$:

(0) an appropriate calculus of propositional logic,
(1) $\forall x\varphi \rightarrow \varphi(x/t)$, if t is free for x in φ,
(2) $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x\psi)$, if there is no free occurrence of x in φ,
(3) $\forall x\, x \doteq x$,
(4) $t_1 \doteq t_2 \rightarrow (\varphi(x/t_1) \rightarrow \varphi(x/t_2))$, if t_1 and t_2 are free for x in φ.

(G1) $\mathrm{G}(\varphi \rightarrow \psi) \rightarrow (\mathrm{G}\varphi \rightarrow \mathrm{G}\psi)$,
(G2) $\mathrm{G}\varphi \rightarrow \mathrm{GG}\varphi$,
(GP) $\varphi \rightarrow \mathrm{GP}\varphi$,
(FG) $\mathrm{F}\varphi \rightarrow \mathrm{G}(\mathrm{F}\varphi \vee \varphi \vee \mathrm{P}\varphi)$,
(∀G) $\forall x\mathrm{G}\varphi \rightarrow \mathrm{G}\forall x\varphi$.

(H1) $\mathrm{H}(\varphi \rightarrow \psi) \rightarrow (\mathrm{H}\varphi \rightarrow \mathrm{H}\psi)$,
(H2) $\mathrm{H}\varphi \rightarrow \mathrm{HH}\varphi$,
(HF) $\varphi \rightarrow \mathrm{HF}\varphi$,
(PH) $\mathrm{P}\varphi \rightarrow \mathrm{H}(\mathrm{F}\varphi \vee \varphi \vee \mathrm{P}\varphi)$,
(∀H) $\forall x\mathrm{H}\varphi \rightarrow \mathrm{H}\forall x\varphi$.

(N1) $\mathrm{N}(\varphi \rightarrow \psi) \rightarrow (\mathrm{N}\varphi \rightarrow \mathrm{N}\psi)$,
(N2) $\mathrm{N}\varphi \rightarrow \mathrm{NN}\varphi$,
(N3) $\mathrm{N}\varphi \rightarrow \varphi$,
(NM) $\varphi \rightarrow \mathrm{NM}\varphi$,
(∀N) $\forall x\mathrm{N}\varphi \rightarrow \mathrm{N}\forall x\varphi$.

(NE) $Ex \rightarrow \mathrm{N}Ex$.

(PN) $\mathrm{PN}\varphi \rightarrow \mathrm{NP}\varphi$ "bridging-principle".

Rules of $\boldsymbol{K}$:

$$\text{(MP)}\ \frac{\varphi, \varphi \to \psi}{\psi},$$

$$(\forall)\ \frac{\varphi}{\forall x \varphi},$$

$$(\mathbf{R}\text{-G})\ \frac{\varphi}{\mathrm{G}\varphi},$$

$$(\mathbf{R}\text{-H})\ \frac{\varphi}{\mathrm{H}\varphi},$$

$$(\mathbf{R}\text{-N})\ \frac{\varphi}{\mathrm{N}\varphi}.$$

The **calculus $\boldsymbol{K}^{\mathrm{irr}}$** is the basic calculus K with the additional rule ("irreflexivity rule"):

$$(\mathbf{R}\text{-IrrN})\ \frac{\exists x \mathrm{N}(Rx \wedge \mathrm{H}\neg Rx) \to \varphi}{\varphi}$$ where R is a unary relation symbol that does not occur in φ.

The **calculus $\boldsymbol{K}_\Box$** is an expansion of the calculus K.[4] We add:

($\Box$1) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
($\Box$2) $\Box\varphi \to \Box\Box\varphi$,
($\Box$3) $\Box\varphi \to \varphi$,
($\Box\Diamond$) $\varphi \to \Box\Diamond\varphi$,
($\forall\Box$) $\forall x \Box\varphi \to \Box\forall x\varphi$.

$$\left.\begin{array}{ll} (\mathrm{F}\Box) & \mathrm{F}\Box\varphi \to \Box\mathrm{F}\varphi, \\ (\mathrm{P}\Box) & \mathrm{P}\Box\varphi \to \Box\mathrm{P}\varphi, \\ (\Box\mathrm{N}) & \Box\varphi \to \mathrm{N}\varphi, \end{array}\right\} \quad \text{"bridging-principles".}$$

$$(\mathbf{R}\text{-}\Box)\ \frac{\varphi}{\Box\varphi}.$$

To obtain the **calculus $\boldsymbol{K}_\Box^{\mathrm{irr}}$**, we add – instead of the irreflexivity rule (R-IrrN) – the rule:

$$(\mathbf{R}\text{-Irr}\Box)\ \frac{\exists x \Box(Rx \wedge \mathrm{H}\neg Rx) \to \varphi}{\varphi}$$ where R is a unary relation symbol that does not occur in φ.

The concepts of **proof, proof of a formula** and **provable formulas** (with respect to the calculus in question) are defined as usual. Given a theory (that is: a set of sentences) Γ and a formula φ, φ is said to be **provable from** Γ iff φ is provable or $\gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$ is provable for some

$\gamma_1, \ldots, \gamma_n \in \Gamma$. If this is the case, we write $\Gamma \vdash \varphi$. A theory Σ is said to be **inconsistent** iff the contradiction $\bot$ is provable from Σ. Otherwise Σ is said to be **consistent**. A consistent theory Σ is said to be **maximal consistent** iff there exists no consistent theory which includes Σ as a proper subset.

In what follows, the symbol $\boxminus$ is used as representing one of the operator symbols out of $\{\mathrm{G}, \mathrm{H}, \mathrm{N}\}$ resp. $\{\mathrm{G}, \mathrm{H}, \mathrm{N}, \Box\}$. Let $\diamondminus$ be defined by $\neg \boxminus \neg$; this means that $\diamondminus$ represents the operator F, P, M, $\Diamond$, respectively. Furthermore, we say that H is the dual of G, G the dual of H, N and $\Box$ the dual of itself. $\boxminus^*$ will denote the dual of $\boxminus$, and $\diamondminus^*$ is the abbreviation of $\neg \boxminus^* \neg$.

Obviously, the usual theorems of predicate, tense and modal logic are provable, especially the following which will be used in the completeness proof:

(**T**-1) $x \doteq y \to y \doteq x$,
(**T**-2) $x \doteq y \to (y \doteq z \to x \doteq z)$,
(**T**-3) $\exists x(\varphi \wedge \psi) \leftrightarrow \varphi \wedge \exists x\psi$, if there is no free occurrence of x in φ,

(**T**-$\boxminus$1) $\boxminus(\varphi \wedge \psi) \leftrightarrow \boxminus\varphi \wedge \boxminus\psi$,
(**T**-$\boxminus$2) $\diamondminus\exists x\varphi \leftrightarrow \exists x\diamondminus\varphi$,
(**T**-$\boxminus$3) $t_1 \doteq t_2 \to \boxminus\, t_1 \doteq t_2$,
(**T**-$\boxminus$4) $t_1 \not\doteq t_2 \to \boxminus\, t_1 \not\doteq t_2$.

The Barcan-formulas ($\forall\boxminus$) are not needed as axioms, since they are provable by axioms (1) and (2), rule ($\forall$) and the rules

$$(\mathbf{R}\text{-}\boxminus\,\diamondminus^*)\ \frac{\varphi \to \boxminus\psi}{\diamondminus^*\varphi \to \psi} \quad \text{and}$$

$$(\mathbf{R}\text{-}\diamondminus^*\boxminus)\ \frac{\diamondminus^*\varphi \to \psi}{\varphi \to \boxminus\psi}$$

which are provable by (R-$\boxminus^*$), ($\boxminus^*$1) and ($\boxminus^*\diamondminus$) resp. (R-$\boxminus$), ($\boxminus$1) and ($\boxminus\diamondminus^*$). In the calculus $K_{\Box}^{\mathrm{irr}}$, the irreflexivity rule (R-IrrN) is provable (use principle ($\Box$N)). Hence, all the $\mathcal{L}$-formulas provable in the calculus K^{irr} are also provable in the calculus $K_{\Box}^{\mathrm{irr}}$. Some further theorems:

(**T**-NG) $\mathrm{NG}\varphi \to \mathrm{GN}\varphi$,
(**T**-HN) $\mathrm{HN}\varphi \to \mathrm{NH}\varphi$.

(**T**-E1) $\forall^a x\varphi \to (Ex \to \varphi)$,
(**T**-E2) $\forall^a xEx$,
(**T**-E3) $\forall^a x\mathrm{N}Ex$,
(**T**-E4) $\neg Ex \to \mathrm{N}\neg Ex$,
(**T**-E5) $\mathrm{P}Ex \to \mathrm{NP}Ex$,

(**T**-E6) $\mathrm{H}Ex \to \mathrm{NH}Ex$,
(**T**-E7) $\mathrm{M}Ex \to \mathrm{GNP}Ex$,
(**T**-E8) $Ex \to \boxminus_1 \dots \boxminus_k\, \mathrm{GNP}Ex$, where $\boxminus_i \in \{\mathrm{G}, \mathrm{N}\}$.

(**T**-N$\forall^a$) $\mathrm{N}\forall^a x\varphi \to \forall^a x\mathrm{N}\varphi$,
(**T**-$\forall^a$N) $\forall^a x\mathrm{N}\varphi \to \mathrm{N}\forall^a x\varphi$,
(**T**-M$\forall^a$) $\mathrm{M}\forall^a x\varphi \to \forall^a x\mathrm{M}\varphi$.

(**T**-□G) $\Box\mathrm{G}\varphi \to \mathrm{G}\Box\varphi$,
(**T**-G□) $\mathrm{G}\Box\varphi \to \Box\mathrm{G}\varphi$,
(**T**-□H) $\Box\mathrm{H}\varphi \to \mathrm{H}\Box\varphi$,
(**T**-H□) $\mathrm{H}\Box\varphi \to \Box\mathrm{H}\varphi$.

3. KAMP- AND T×W-STRUCTURES

DEFINITION 3.1. A **Kamp-frame** is a quadruple $\mathfrak{R} = (T, W, <, \sim)$ consisting of:

(a) a nonempty set T (“moments”);
(b) a nonempty set W (“worlds”, “histories”);
(c) a map $<$ which assigns to each $w \in W$ a linear ordering $(T_w, <_w)$ with $\bigcup_{w\in W} T_w = T$;
(d) a map $\sim$ which assigns to each $t \in T$ an equivalence relation $\sim_t$ on the set $\{w \in W \mid t \in T_w\}$ such that the following condition holds:

$$w \sim_{t_2} w' \quad \text{and} \quad t_1 <_w t_2 \Rightarrow w \sim_{t_1} w' \quad \text{and} \quad t_1 <_{w'} t_2,$$

for all $w, w' \in W$, $t_1, t_2 \in T$.

A **Kamp-structure** is a quintuple $\mathfrak{A} = (\mathfrak{R}, U, e, k, p)$ consisting of

(a) A Kamp-frame $\mathfrak{R} = (T, W, <, \sim)$;
(b) a nonempty set U (“individuals”, “possible objects”);
(c) a map e that assigns to each $t \in T$ and each $w \in W$ with $t \in T_w$ a set $U_{t,w} \subseteq U$ such that

$$(*) \qquad w \sim_t w' \Rightarrow U_{t,w} = U_{t,w'},$$

for all $w, w' \in W$, $t \in T$;
(d) a map k that assigns to each constant c an individual $k(c) \in U$;
(e) a map p that assigns to each m-ary relation symbol R at every moment $t \in T$ and every $w \in W$ with $t \in T_w$ a set $p_{t,w}(R) \subseteq U^m$.

Note that from the definition of a Kamp-frame we obtain: $t_1 <_w t_2 \Rightarrow t_1, t_2 \in T_w$ and $w \sim_t w' \Rightarrow t \in T_w \cap T_{w'}$. A moment t is said to be **in a history** w iff $t \in T_w$. Read “$w \sim_t w'$” as: w and w' share up to (and inclusively) t the same history. $U_{t,w}$ is said to be **the set of all objects existing at** t **in** w. These readings justify condition $(*)$.[5]

A special kind of Kamp-frames are those Kamp-frames which satisfy the condition

$$w \sim_t w' \quad \text{iff} \quad t \in T_w \cap T_{w'}.$$

From the completeness result in Section 6 one can conclude that it is not possible to characterize this condition by any $\mathcal{L}$-formula.

DEFINITION 3.2. A **T×W-frame** is a quadruple $\mathfrak{R} = (T, <, W, \sim)$ consisting of:

(a) a nonempty set T (“time-points”),
(b) a linear order $<$ on T,
(c) a nonempty set W (“histories”, “worlds”),
(d) a family $\sim = (\sim_t)_{t \in T}$ of equivalence relations $\sim_t$ on W such that the following condition is satisfied:

$$w \sim_{t_2} w' \quad \text{and} \quad t_1 < t_2 \Rightarrow w \sim_{t_1} w',$$

for all $w, w' \in W, t_1, t_2 \in T$.

A **T×W-structure** is a quintuple $\mathfrak{A} = (\mathfrak{R}, U, e, k, p)$ consisting of

(a) a T×W-frame $\mathfrak{R} = (T, <, W, \sim)$;
(b) a nonempty set U (“individuals”, “possible objects”);
(c) a map e that assigns to each $t \in T$ and each $w \in W$ a set $U_{t,w} \subseteq U$ such that

$$w \sim_t w' \Rightarrow U_{t,w} = U_{t,w'},$$

for all $w, w' \in W, t \in T$;
(d) a map k which assigns to each constant c an individual $k(c) \in U$;
(e) a map p which assigns to each m-ary relation symbol R at every time-point $t \in T$ and every world $w \in W$ a set $p_{t,w}(R) \subseteq U^m$.

In what follows, by “frame” or “structure” we understand Kamp-frames resp. Kamp-structures, or T×W-frames resp. T×W-structures depending on the prevailing context. If $\mathfrak{A} = (\mathfrak{R}, U, \ldots)$ is a structure, we will denote the “domain” U by $|\mathfrak{A}|$.

DEFINITION 3.3. A **(variable-) assignment in a structure** $\mathfrak{A}$ is a map h which assigns to every variable x an individual $h(x) \in |\mathfrak{A}|$. The **value of a term** t **at an assignment** h **in** $\mathfrak{A}$ ist defined by

$$|t|_h^{\mathfrak{A}} := \begin{cases} h(t), & \text{if } t \text{ is a variable;} \\ k(t), & \text{if } t \text{ is a constant.} \end{cases}$$

Given an assignment h in $\mathfrak{A}$, a fixed variable x and a fixed individual $\mathfrak{a} \in |\mathfrak{A}|$, the map $h^x_\mathfrak{a}$ defined by

$$h^x_\mathfrak{a}(y) := \begin{cases} \mathfrak{a}, & \text{if } y = x, \\ h(y), & \text{else} \end{cases}$$

is an assignment in $\mathfrak{A}$, too.

We recursively define the concept of a $\mathcal{L}_\Box$**-formula being valid at an assignment** h **in a T×W-structure** $\mathfrak{A}$ **at a time-point** $t \in T$ **and a history** $w \in W$:

$$\begin{array}{lll}
\mathfrak{A}, h \models_{t,w} Es & \text{iff} & |s|^\mathfrak{A}_h \in U_{t,w}, \\
\mathfrak{A}, h \models_{t,w} t_1 \doteq t_2 & \text{iff} & |t_1|^\mathfrak{A}_h = |t_2|^\mathfrak{A}_h, \\
\mathfrak{A}, h \models_{t,w} R(t_1, \ldots, t_m) & \text{iff} & (|t_1|^\mathfrak{A}_h, \ldots, |t_m|^\mathfrak{A}_h) \in p_{t,w}(R), \\
\mathfrak{A}, h \models_{t,w} \neg\varphi & \text{iff} & \mathfrak{A}, h \not\models_{t,w} \varphi, \\
\mathfrak{A}, h \models_{t,w} \varphi \to \psi & \text{iff} & \mathfrak{A}, h \not\models_{t,w} \varphi \text{ or } \mathfrak{A}, h \models_{t,w} \psi, \\
\mathfrak{A}, h \models_{t,w} \mathrm{G}\varphi & \text{iff} & \mathfrak{A}, h \models_{t',w} \varphi, \text{ for all } t' > t, \\
\mathfrak{A}, h \models_{t,w} \mathrm{H}\varphi & \text{iff} & \mathfrak{A}, h \models_{t',w} \varphi, \text{ for all } t' < t, \\
\mathfrak{A}, h \models_{t,w} \mathrm{N}\varphi & \text{iff} & \mathfrak{A}, h \models_{t,w'} \varphi, \text{ for all } w' \sim_t w, \\
\mathfrak{A}, h \models_{t,w} \Box\varphi & \text{iff} & \mathfrak{A}, h \models_{t,w'} \varphi, \text{ for all } w' \in W, \\
\mathfrak{A}, h \models_{t,w} \forall x\varphi & \text{iff} & \mathfrak{A}, h^x_\mathfrak{a} \models_{t,w} \varphi, \text{ for all } \mathfrak{a} \in U.
\end{array}$$

A formula φ is said to be **valid in a T×W-structure** $\mathfrak{A}$ **at** $t \in T$ **and** $w \in W$ iff φ is valid at every assignment h in $\mathfrak{A}$ at t and w. φ is **valid in** $\mathfrak{A}$ iff φ is valid in $\mathfrak{A}$ at every $t \in T$ and every $w \in W$. $\mathfrak{A}$ **satisfies or models a sentence** σ **at** $t \in T$ **and** $w \in W$ iff σ is valid in $\mathfrak{A}$ at t and w. $\mathfrak{A}$ **satisfies or models a set of sentences** Σ **at** $t \in T$ **and** $w \in W$ iff $\mathfrak{A}$ satisfies every $\sigma \in \Sigma$ at t and w. $\mathfrak{A}$ **satisfies** Σ iff there are $t \in T$ and $w \in W$ such that $\mathfrak{A}$ satisfies Σ at t and w.

Analogously, the concept of a $\mathcal{L}$**-formula being valid at an assignment** h **in a Kamp-structure** $\mathfrak{A}$ **at** $t \in T$ **and** $w \in W$ is defined; note that with respect to Kamp-structures, "$\models_{t,w}$" is only defined in case $t \in T_w$. And obviously, the clause for $\Box$ has to be dropped.

As in ordinary predicate logic, we can prove:

THEOREM 3.4 (Coincidence-Theorem). *Let* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *be Kamp-* (*resp.* T×W-) *structures, both defined on the same frame* $\mathfrak{R}$ *and the same domain* U. *Moreover, let* h *and* h' *be assignments, the first in* $\mathfrak{A}$ *and the second in* $\mathfrak{A}'$. *Now, let* φ *be a* $\mathcal{L}$- (*resp.* $\mathcal{L}_\Box$-) *formula such that the following conditions hold*:

(a) $\mathfrak{A}$ *and* $\mathfrak{A}'$ *agree on the interpretation of all constants and relation symbols that occur in* φ;

(b) h *and* h' *agree on the interpretation of all variables that occur free in* φ.

Then, we have the following equivalence:

$$\mathfrak{A}, h \models_{t,w} \varphi \quad \textit{iff} \quad \mathfrak{A}', h' \models_{t,w} \varphi,$$

for every $t \in T$ *and* $w \in W$ *with* $t \in T_w$ (*resp.: for every* $t \in T$ *and* $w \in W$).

THEOREM 3.5 (Substitution-Theorem). *Let* φ *be a formula,* x *a variable,* s *a term which is free for* x *in* φ *and* $\mathfrak{A}$ *a structure. Then for every* $t \in T$, $w \in W$ *and every assignment* h *in* $\mathfrak{A}$*, the following equivalence holds:*

$$\mathfrak{A}, h \models_{t,w} \varphi(x/s) \quad \textit{iff} \quad \mathfrak{A}, h^{x}_{|s|^{\mathfrak{A}}_{h}} \models_{t,w} \varphi.$$

From these theorems we obtain that the calculus K^{irr} is sound with respect to Kamp-semantics and that the calculus $K^{\mathrm{irr}}_{\Box}$ is sound with respect to T×W-semantics:

THEOREM 3.6 (Soundness of K^{irr} and $K^{\mathrm{irr}}_{\Box}$). *Any* $\mathcal{L}$*-formula provable in the calculus* K^{irr} *is valid in each Kamp-structure. Any* $\mathcal{L}_{\Box}$*-formula provable in the calculus* $K^{\mathrm{irr}}_{\Box}$ *is valid in each* T×W*-structure.*

Moreover: Every set of $\mathcal{L}$*-sentences which is satisfied by a Kamp-structure is consistent with respect to* K^{irr}*. Every set of* $\mathcal{L}_{\Box}$*-sentences which is satisfied by a* T×W*-structure is consistent with respect to* $K^{\mathrm{irr}}_{\Box}$.

Proof. We only show that the rule (R-Irr$\Box$) preserves T×W-validity. Let φ be a formula and R be a unary relation symbol that does not occur in φ. Assume that there are a T×W-structure $\mathfrak{A} = (\mathfrak{R}, U, e, k, p)$, an assignment h, $t_0 \in T$ and $w_0 \in W$ with $\mathfrak{A}, h \not\models_{t_0,w_0} \varphi$. We set $p'_{t,w}(R') := p_{t,w}(R')$, for every relation symbol R' different from R, and further, for a fixed $\mathfrak{a} \in U$:

$$p'_{t,w}(R) := \begin{cases} p_{t,w}(R) \setminus \{\mathfrak{a}\} & \text{if } t < t_0, \\ p_{t,w}(R) \cup \{\mathfrak{a}\} & \text{else.} \end{cases}$$

By the Coincidence-Theorem and the irreflexivity of $<$, it can easily be shown that the formula $\exists x \Box (Rx \wedge \mathrm{H}\neg Rx) \rightarrow \varphi$ is not valid in the structure $\mathfrak{A}' := (\mathfrak{R}, U, e, k, p')$ at t_0 and w_0. $\Box$

Some further remarks may be in order: A frame is said to be **modally connected** iff for all $w, w' \in W$, there is a $t \in T$ with $w \sim_t w'$. A structure is **modally connected** iff its frame is so. A structure is said to be **total** iff $U = \bigcup_{t,w} U_{t,w}$. A linear order (X, R) is said to be **infinite to the left** iff every $x \in X$ has an R-predecessor. A T×W-frame (or -structure) is **infinite in the past** iff $(T, <)$ is infinite to the left. A Kamp-frame (or

-structure) is **infinite in the past** iff for every w, the linear order $(T_w, <_w)$ is infinite to the left. Note that a modally connected Kamp-frames is infinite in the past if $(T_w, <_w)$ is infinite to the left, for at least one world w.

In ordinary tense logic, infinity in the past is axiomatizable by

(1) $\mathrm{H}\varphi \rightarrow \mathrm{P}\varphi$ or equivalently: $\mathrm{P}\top$.

Modal connectedness cannot be axiomatized in the language $\mathcal{L}$ – this follows from the completeness result in Section 6. But in the language $\mathcal{L}_\Box$ this property can be expressed by the formula:[6]

(2) $\Diamond\varphi \rightarrow \mathrm{PMF}\varphi$

if we assume infinity in the past. With respect to T×W-semantics totality can be axiomatized by

(3) $\Diamond(Ex \vee \mathrm{P}Ex \vee \mathrm{F}Ex)$

and with respect to Kamp-semantics by

(4) $Ex \vee \mathrm{MF}Ex \vee \mathrm{PMF}Ex.$

From (4) and (1) (resp.: from (3), (1) and (2)), we can derive

(5) $\mathrm{PMF}Ex$

– use (A-HF), (T-NG), (T-G$\Box$), (H2) and (G2). Indeed, this formula axiomatizes “totality and infinity in the past” with respect to Kamp-semantics (cf. Section 6), and expresses this property with respect to modally connected T×W-frames.

4. IRREFLEXIVE THEORIES

We next introduce the concept of “irreflexive theory”: A sentence σ is said to be **modaltypical** iff σ has the form

$$⟠_1(\varphi_1 \wedge \ldots ⟠_{n-1}(\varphi_{n-1} \wedge ⟠_n\varphi_n) \ldots),$$

where $\varphi_1, \ldots, \varphi_n$ are $\mathcal{L}$-formulas (resp. $\mathcal{L}_\Box$-formulas) and $⟠_1, \ldots, ⟠_n \in \{\mathrm{F}, \mathrm{P}, \mathrm{M}\}$ (resp. $⟠_1, \ldots, ⟠_n \in \{\mathrm{F}, \mathrm{P}, \mathrm{M}, \Diamond\}$). In the following considerations let $\chi(R, x)$ denote the formula $\mathrm{N}(Rx \wedge \mathrm{H}\neg Rx)$ or $\Box(Rx \wedge \mathrm{H}\neg Rx)$, respectively.

DEFINITION 4.1. A theory Σ **has the $\exists$-property** iff for every existential sentence $\exists x\varphi \in \Sigma$ there is a constant c with $\varphi(x/c) \in \Sigma$.[7]

A theory Σ **has the** Irr-**property** iff for every modaltypical sentence

$$\Phi(\varphi_n) := ⟠_1(\varphi_1 \wedge \ldots ⟠_{n-1}(\varphi_{n-1} \wedge ⟠_n\varphi_n) \ldots)$$

in Σ, there is a unary relation symbol R such that R does not occur in any of the sentences $\varphi_1, \ldots, \varphi_n$ and Σ contains

$$\begin{aligned}\Phi(\varphi_n \wedge \exists x \chi(R, x)) \;:=\; & \diamondminus_1(\varphi_1 \wedge \ldots \diamondminus_{n-1}(\varphi_{n-1} \wedge \\ & \wedge \diamondminus_n(\varphi_n \wedge \exists x \chi(R, x))) \ldots).^{8}\end{aligned}$$

A theory Σ is said to be **irreflexive** iff (a) Σ is maximal consistent, (b) Σ has the $\exists$- and the Irr-property, and (c) there is unary relation symbol R with $\exists x \chi(R, x) \in \Sigma$.

We will show that on some weak conditions each consistent theory is includeable in an irreflexive theory. To prove this we will need the following propositions:

PROPOSITION 4.2. *Let* Σ *be a theory,* $\exists x \varphi$ *a sentence and* c *a constant that does not occur either in* Σ *or in* φ. *If the theory* $\Sigma \cup \{\exists x \varphi\}$ *is consistent, then so is the theory* $\Sigma \cup \{\exists x \varphi, \varphi(x/c)\}$.

Proof. As in ordinary predicate logic, using the following fact: If $\varphi_1, \ldots,$ φ_n is a proof (in K^{irr} resp. $K^{\mathrm{irr}}_{\Box}$), and if y is a variable which does not occur in any φ_i, then the sequence

$$\varphi_1(c/y), \ldots, \varphi_n(c/y)$$

is a proof in which the constant c does not occur. □

In analogy to Proposition 4.2, we have:

PROPOSITION 4.3. *Let* Σ *be a theory,* $\Phi(\tau)$ *a modaltypical sentence and* R *a unary relation symbol that occurs neither in* Σ *nor in* $\Phi(\tau)$. *Now, if the theory* $\Sigma \cup \{\Phi(\tau)\}$ *is consistent, then so is the theory*

$$\Sigma \cup \{\Phi(\tau), \Phi(\tau \wedge \exists x \chi(R, x))\}.$$

Proof. For the proof use Theorem 3.2.4 in Gabbay et al. (1994), pp. 93–94. □

We are now able to establish essential theorems about irreflexive theories (for the constructions, cf. Gabbay et al. (1994), pp. 182ff):

THEOREM 4.4. *Every consistent theory* Σ *in which infinitely many constants and infinitely many unary relation symbols do not occur can be included in an irreflexive theory.*

Proof. Let $\varphi_0, \varphi_1, \ldots$ be an enumeration of all sentences. We define an ascending sequence of theories

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$$

such that each Σ_i has the following properties:

(a) Σ_i is consistent.
(b) There are infinitely many constants and unary relation symbols that do not occur in Σ_i.

First, we choose an arbitrary relation symbol R that does not occur in Σ. By the irreflexivity rule, it can easily be shown that the theory $\Sigma_0 := \Sigma \cup \{\exists x \chi(R, x)\}$ is consistent. Obviously, Σ_0 also satisfies condition (b).

We suppose now that Σ_n has been defined and satisfies both of the above conditions. We set:

$$\Sigma_{n+1} := \Sigma_n \cup \begin{cases} \{\neg\varphi_n\} & \text{if } \Sigma_n \cup \{\neg\varphi_n\} \text{ is consistent,} \\ \{\exists x\psi, \psi(x/c_k)\} & \text{if } \Sigma_n \cup \{\neg\varphi_n\} \text{ is inconsis-} \\ & \text{tent and } \varphi_n = \exists x\psi, \\ \{\Phi(\tau), \Phi(\tau \wedge \exists x \chi(R_l, x))\} & \text{if } \Sigma_n \cup \{\neg\varphi_n\} \text{ is inconsis-} \\ & \text{tent and } \varphi_n = \Phi(\tau), \\ \{\varphi_n\} & \text{else} \end{cases}$$

– in the second case, choose the constant with the lowest index k such that c_k does not occur in $\Sigma_n \cup \{\varphi_n\}$, and in the third case the unary relation symbol with the lowest index l which occurs neither in Σ_n nor in $\Phi(\tau)$. By Propositions 4.2 and 4.3, the theory Σ_{n+1} is consistent. Then the theory $\Sigma^\infty := \bigcup_{k \in \mathbb{N}} \Sigma_k$ is irreflexive, by construction. □

We will see that Theorem 4.4 suffices to show "weak completeness" of the calculi in question in case that $\mathcal{L}_{(\Box)}$ contains denumerably many constants. In order to achieve a better result, we are going to show that each consistent theory can be included in an irreflexive theory of some richer language. Let $\mathcal{L}^*$ resp. $\mathcal{L}^*_\Box$ denote the languages which are obtained from $\mathcal{L}$ resp. $\mathcal{L}_\Box$ by introducing denumerably many new individual constants

$$c_1^*, c_2^*, \ldots$$

and denumerably many new unary relation symbols

$$R_1^*, R_2^*, \ldots .$$

Obviously, every theory of $\mathcal{L}_{(\Box)}$ is a theory in the language $\mathcal{L}^*_{(\Box)}$. In what follows we have to show that every consistent $\mathcal{L}_{(\Box)}$-theory is a consistent $\mathcal{L}^*_{(\Box)}$-theory – where Σ is said to be a **consistent L-theory** iff (a) Σ is a set of L-sentences, and (b) there is no finite sequence of L-formulas which shows Σ to be inconsistent (with respect to the calculus in question). We have to show:

(1) If Σ is a consistent $\mathcal{L}_{(\Box)}$-theory, then Σ is a consistent $\mathcal{L}'_{(\Box)}$-theory, where $\mathcal{L}'_{(\Box)}$ denotes the language $\mathcal{L}_{(\Box)} \cup \{c_1^*, c_2^*, \ldots\}$.
(2) If Σ' is a consistent $\mathcal{L}'_{(\Box)}$-theory, then Σ' is a consistent $\mathcal{L}^*_{(\Box)}$-theory. (Note that $\mathcal{L}^*_{(\Box)} = \mathcal{L}'_{(\Box)} \cup \{R_1^*, R_2^*, \ldots\}$.)

The first claim is a trivial extension of a standard theorem in quantified modal logic. Essentially, it depends on the remark made in the proof of Proposition 4.2. For the second claim the procedure is analogous: Let L denote an extension of $\mathcal{L}'_{(\Box)}$ which differs from $\mathcal{L}'_{(\Box)}$ in not more than some new unary relation-symbols. (Note that the language L is such that the existence of denumerably many unary relation symbols is guaranteed.) Then the second claim can be concluded from the following propositions:

(a) If the sequence $\varphi_1, \dots, \varphi_n$ of $L \cup \{S\}$-formulas (where S is a new unary relation symbol) is a proof with respect to $K^{\mathrm{irr}}_{(\Box)}$, then there is a unary relation symbol R of L such that the sequence $\varphi_1(S/R), \dots, \varphi_n(S/R)$ of L-formulas is a proof with respect to that calculus.[9]

From (a) it follows immediately:

(b) Each consistent L-theory is a consistent $L \cup \{S\}$-theory, where S is a new unary relation symbol.

By induction, we obtain:

(c) Each consistent L-theory is a consistent L'-theory, where L' is a finite extension of L of the form $L \cup \{S_1, \dots, S_k\}$, for unary relation symbols $S_1, \dots, S_k$.

Suppose now Σ' to be a theory of the language $\mathcal{L}'_{(\Box)}$ which is inconsistent considered as a theory of the language $\mathcal{L}^*_{(\Box)} = \mathcal{L}'_{(\Box)} \cup \{R^*_1, R^*_2, \dots\}$. Then there are $\sigma'_1, \dots, \sigma'_m \in \Sigma'$ with $\vdash \sigma'_1 \wedge \dots \wedge \sigma'_m \rightarrow \bot$. Hence, there is a proof sequence $\varphi^*_1, \dots, \varphi^*_n$ of $\mathcal{L}^*_{(\Box)}$-formulas with $\varphi^*_n = \sigma'_1 \wedge \dots \wedge \sigma'_m \rightarrow \bot$. In this sequence, only finitely many of the new symbols R^*_i occur, say $R^*_{i_1}, \dots, R^*_{i_l}$. Suppose that i_k is the largest index in $\{i_1, \dots, i_l\}$. Then the sequence $\varphi^*_1, \dots, \varphi^*_n$ is a proof in the language $\mathcal{L}' \cup \{R^*_1, \dots, R^*_{i_k}\}$. Thus, Σ is inconsistent considered as a $\mathcal{L}' \cup \{R^*_1, \dots, R^*_{i_k}\}$-theory, and hence by (c), inconsistent as a $\mathcal{L}'$-theory.

The crucial point is the proof of (a): Let $\varphi_1, \dots, \varphi_n$ be a sequence of $L \cup \{S\}$-formulas which is a proof with respect to $K^{\mathrm{irr}}_{(\Box)}$. Assume that the claim has been shown for all proof sequences of length $< n$. Choose a unary relation symbol R of L which does not occur in any of the formulas φ_i. If φ_n is an axiom of the calculus $K^{\mathrm{irr}}_{(\Box)}$, $\varphi_n(S/R)$ is an axiom, too. If φ_n results from φ_i by application of the irreflexivity rule – all other rules are handled in a similar way –, then φ is of the form $\exists x \chi(R', x) \rightarrow \varphi_n$, where R' is a unary relation symbol that does not occur in φ_n. If $S = R'$, we obtain $\varphi_i(S/R) = \exists x \chi(R, x) \rightarrow \varphi_n(S/R)$. In this case R does not occur in $\varphi_n(S/R)$, since R and S do not occur in φ_n. If S is different from R', we get $\varphi_i(S/R) = \exists x \chi(R', x) \rightarrow \varphi_n(S/R)$. Here R' does not occur in

$\varphi_n(S/R)$. Therefore $\varphi_n(S/R)$ is derivable from $\varphi_i(S/R)$ by application of the irreflexivity rule.

COROLLARY 4.5. *Every consistent $\mathcal{L}_{(\Box)}$-theory is included in an irreflexive theory of the language $\mathcal{L}^*_{(\Box)}$.*

Given a theory Γ, we define:

$$\boxminus(\Gamma) := \{\gamma \mid \boxminus\gamma \in \Gamma\}.$$

LEMMA 4.6. *If a sentence $\diamondminus\varphi$ is consistent with a theory Γ, then $\boxminus(\Gamma) \cup \{\varphi\}$ is consistent.*

THEOREM 4.7. *If Σ is an irreflexive theory and ρ is a sentence with $\diamondminus\rho \in \Sigma$, then there is an irreflexive theory Γ which contains ρ as an element and includes the theory $\boxminus(\Sigma)$.*

Proof. Let $\varphi_0, \varphi_1, \ldots$ be an enumeration of all sentences. We define an ascending sequence of consistent theories $\boxminus(\Sigma) \cup \{\rho\} \subseteq \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ as in the proof of Lemma 6.2.5 in Gabbay et al. (1994), pp. 185–186. We only have to consider the following case:

(a) Γ_n is already defined and consistent,
(b) there are only finitely many sentences in Γ_n which are not in $\boxminus(\Sigma)$,
(c) $\neg\varphi_n$ is inconsistent with Γ_n, and
(d) φ_n is an existential sentence of the form $\exists x\psi$.

In this case, Σ contains the sentence

$$(*) \qquad \diamondminus(\gamma_n \wedge \varphi_n),$$

where γ_n is the conjunction of all the (finitely many) sentences in Γ_n not in $\boxminus(\Sigma)$. This can be seen as follows: Suppose $(*)$ to be not in Σ. Then, by the maximality of Σ, the sentence $\boxminus(\gamma_n \rightarrow \neg\varphi_n)$ is in Σ, hence the sentence $\gamma_n \rightarrow \neg\varphi_n$ is in $\boxminus(\Sigma)$ and therefore $\Gamma_n \vdash \neg\varphi_n$. Thus we obtain that $\Gamma_n \cup \{\varphi_n\}$ is inconsistent – a contradiction. From $\diamondminus(\gamma_n \wedge \exists x\psi) \in \Sigma$ it follows $\diamondminus\exists x(\gamma_n \wedge \psi) \in \Sigma$ – the variable x does not occur free in γ_n –, and hence, by Theorem (T-$\boxminus$2), $\exists x\diamondminus(\gamma_n \wedge \psi) \in \Sigma$. Now, since Σ has the $\exists$-property, there is a constant c with $\diamondminus(\gamma_n \wedge \psi)(x/c) \in \Sigma$. From this we get $\diamondminus(\gamma_n \wedge \psi(x/c)) \in \Sigma$, and we set

$$\Gamma_{n+1} := \Gamma_n \cup \{\exists x\psi, \psi(x/c)\}.$$

This theory satisfies condition (a) by Lemma 4.6, and obviously, condition (b). □

5. EQUIVALENCE RELATIONS

Given maximal consistent theories Σ and Σ', we define as usual:

$$\Sigma R_{\boxminus} \Sigma' \quad \text{iff} \quad \boxminus(\Sigma) \subseteq \Sigma',$$

where $\boxminus \in \{\mathrm{G}, \mathrm{H}, \mathrm{N}, \Box\}$. Obviously, $\Sigma R_{\mathrm{G}} \Sigma'$ holds if and only if $\Sigma' R_{\mathrm{H}} \Sigma$ holds. Each of the relations R_{N} and $R_{\Box}$ is reflexive, symmetric and transitive; the relation R_{G} is transitive, left- and right-linear. More importantly:

LEMMA 5.1. *The restriction of R_{G} to an arbitrary set of irreflexive theories is an irreflexive relation.*

Proof. For the proof use condition (c) of Definition 4.1. □

On the set of all irreflexive theories we introduce a binary relation by

$$\Sigma \sim_{\mathrm{G}} \Sigma' \; : \text{iff} \quad \Sigma R_{\mathrm{G}} \Sigma' \quad \text{or} \quad \Sigma = \Sigma' \quad \text{or} \quad \Sigma' R_{\mathrm{G}} \Sigma.$$

$\sim_{\mathrm{G}}$ is an equivalence relation: reflexivity and symmetry are obvious. Its transitivity follows from the transitivity, left- and right-linearity of the relation R_{G}. Thus we can summarize:

PROPOSITION 5.2. *The relation R_{G} is a linear order on every equivalence class modulo $\sim_{\mathrm{G}}$.*

The content of the following simple lemma is an identity-criteria for irreflexive theories which we will use frequently.

LEMMA 5.3. *Let Σ and Σ' be irreflexive theories with $\Sigma \sim_{\mathrm{G}} \Sigma'$. Now, if there is a unary relation symbol R and a constant c with*

$$Rc \wedge \mathrm{H}\neg Rc \in \Sigma \cap \Sigma',$$

then Σ and Σ' are identical.

Proof. The claim follows immediately from condition (c) of Definition 4.1. □

PROPOSITION 5.4.

(a) *Let Σ and Σ' be irreflexive theories with $\Sigma R_{\mathrm{N}} \Sigma'$. Then for every irreflexive theory Γ with $\Gamma R_{\mathrm{G}} \Sigma$, there is exactly one irreflexive theory Γ' with $\Gamma' R_{\mathrm{G}} \Sigma'$ and $\Gamma R_{\mathrm{N}} \Gamma'$.*

(b) *Let Σ and Σ' be irreflexive theories with $\Sigma R_{\Box} \Sigma'$. Then for every irreflexive theory Γ, the following statements holds*:

  (α) *Σ' is the only irreflexive theory in the equivalence class of Σ' modulo $\sim_{\mathrm{G}}$ which is $R_{\Box}$-related with Σ.*

(β) *If* $\Gamma R_{\mathrm{G}} \Sigma$*, then there is exactly one irreflexive theory* Γ' *with* $\Gamma' R_{\mathrm{G}} \Sigma'$ *and* $\Gamma R_{\Box} \Gamma'$.

(γ) *If* $\Sigma R_{\mathrm{G}} \Gamma$*, then there is exactly one irreflexive theory* Γ' *with* $\Sigma' R_{\mathrm{G}} \Gamma'$ *and* $\Gamma R_{\Box} \Gamma'$.

Proof. (a): By the irreflexivity of Γ, there is a unary relation symbol R and a constant c with $\mathrm{N}(Rc \wedge \mathrm{H}\neg Rc) \in \Gamma$. From $\Gamma R_{\mathrm{G}} \Sigma$ we obtain $\mathrm{PN}(Rc \wedge \mathrm{H}\neg Rc) \in \Sigma$, and hence by (PN), $\mathrm{NP}(Rc \wedge \mathrm{H}\neg Rc) \in \Sigma$. Since $\Sigma R_{\mathrm{N}} \Sigma'$ holds, we can conclude: $\mathrm{P}(Rc \wedge \mathrm{H}\neg Rc) \in \Sigma'$. By Theorem 4.7, we obtain an irreflexive theory Γ' with $\Gamma' R_{\mathrm{G}} \Sigma'$ and $Rc \wedge \mathrm{H}\neg Rc \in \Gamma'$. Furthermore, $\Gamma R_{\mathrm{N}} \Gamma'$ holds, since: If $\mathrm{N}\varphi \in \Gamma$, then $\mathrm{N}(\varphi \wedge Rc \wedge \mathrm{H}\neg Rc) \in \Gamma$, hence $\mathrm{PN}(\varphi \wedge Rc \wedge \mathrm{H}\neg Rc) \in \Sigma$, hence $\mathrm{NP}(\varphi \wedge Rc \wedge \mathrm{H}\neg Rc) \in \Sigma$, and finally $\mathrm{P}(\varphi \wedge Rc \wedge \mathrm{H}\neg Rc) \in \Sigma'$. Again, we obtain an irreflexive theory Γ^* with $\Gamma^* R_{\mathrm{G}} \Sigma'$ and $\varphi \wedge Rc \wedge \mathrm{H}\neg Rc \in \Gamma^*$. From this by left-linearity: $\Gamma' \sim_{\mathrm{G}} \Gamma^*$. But each of Γ' and Γ^* contains the sentence $Rc \wedge \mathrm{H}\neg Rc$, consequently both theories are identical (Lemma 5.3). Therefore, Γ' contains φ, establishing the assertion.

Thus, the existence of a theory Γ' with the desired properties has been proved. Its uniqueness is a consequence of Lemma 5.3.

(b): (α) follows immediately from Lemma 5.3. The proof of (β) and (γ) is nearly the same as in (a): Instead of (PN), use (P$\Box$) in the proof of (β), and (F$\Box$) in that of (γ). $\Box$

On the set of all irreflexive theories we define a further equivalence relation by:

$$\begin{aligned} \Sigma \simeq \Sigma' \quad &\text{:iff} \quad \text{for all constants } c \text{ and } d, \\ &c \doteq d \in \Sigma \quad \text{iff} \quad c \doteq d \in \Sigma'. \end{aligned}$$

PROPOSITION 5.5. *If* Σ *and* Σ' *are irreflexive theories with* $\Sigma R_{\boxminus} \Sigma'$*, then both theories are within the same equivalence class modulo* $\simeq$.

Proof. Theorems (T-$\boxminus$3) and (T-$\boxminus$4) are used here. $\Box$

Hence, for each irreflexive theory Σ, the equivalence classes of Σ modulo each of the relations $\sim_{\mathrm{G}}$, R_{N} and $R_{\Box}$ are included in the equivalence class of Σ modulo $\simeq$.

6. CANONICAL KAMP-STRUCTURES AND COMPLETENESS

Given an irreflexive theory Σ and an equivalence relation $\sim$ – defined for irreflexive theories –, let $[\Sigma]_{\sim}$ denote the equivalence class of Σ modulo

$\sim$. At first, we set:

$$[\Sigma]_{\sim_G} \approx [\Sigma']_{\sim_G} \quad \text{:iff} \quad \text{there are } \Gamma, \Gamma' \text{ with } \Gamma' \sim_G \Sigma, \Gamma' \sim_G \Sigma' \text{ and } \Gamma R_N \Gamma'.$$

Obviously, the relation $\approx$ is an equivalence relation, and the union of each equivalence class modulo $\approx$ is included in exactly one equivalence class modulo $\simeq$.

We are now going to define the canonical Kamp-frame with respect to a fixed irreflexive theory Σ_0. Let Ω_0 denote the union of all equivalence classes modulo $\sim_G$ which are $\approx$-related to $[\Sigma]_{\sim_G}$. Obviously, for every equivalence class modulo each of the relations $\sim_G$ and R_N we have: either it is included in Ω_0 or it is disjoint with Ω_0. Let $\Psi, \Psi', \Psi_1, \ldots$ denote equivalence classes modulo $\sim_G$ which are included in Ω_0, and if $\Sigma \in \Omega_0$, let Ψ_Σ denote that Ψ with $\Sigma \in \Psi$. Analogously, let $\Theta, \Theta_\Sigma, \Theta', \Theta_1, \ldots$ denote equivalence classes modulo R_N which are included in Ω_0.

By Lemma 5.3 we obtain: If $\Theta \cap \Psi \neq \emptyset$, then there is exactly one $\Sigma \in \Theta \cap \Psi$. Let Θ/Ψ denote that unique irreflexive theory which lies in both Θ and Ψ. We set:

$$\begin{aligned}
T &:= \text{set of all equivalence classes } \Theta \text{ modulo } R_N \text{ which are included in } \Omega_0,\\
W &:= \text{set of all equivalence classes } \Psi \text{ modulo } \sim_G \text{ which are included in } \Omega_0,\\
T_\Psi &:= \{\Theta \mid \Theta \cap \Psi \neq \emptyset\},\\
\Theta_1 <_\Psi \Theta_2 &\text{ :iff } \Theta_1/\Psi \; R_G \; \Theta_2/\Psi \quad (\Theta_1, \Theta_2 \in T_\Psi),\\
\Psi \sim_\Theta \Psi' &\text{ :iff } \Theta \in T_\Psi \cap T_{\Psi'}.
\end{aligned}$$

Hence, for each equivalence class Ψ the pair $(T_\Psi, <_\Psi)$ is a linear order, and $\sim_\Theta$ is an equivalence relation on the set $\{\Psi \in W \mid \Theta \in T_\Psi\}$. Let $<$ denote the map that assigns to each $\Psi \in W$ the linear order $(T_\Psi, <_\Psi)$.

Furthermore, we have:

PROPOSITION 6.1. *From* $\Psi \sim_{\Theta_1} \Psi'$ *and* $\Theta_2 <_\Psi \Theta_1$ *it follows*

$$\Psi \sim_{\Theta_2} \Psi' \quad \textit{and} \quad \Theta_2 <_{\Psi'} \Theta_1.$$

Proof. By definition, we have $\Theta_1 \in T_\Psi \cap T_{\Psi'}$ and $\Theta_2/\Psi \; R_G \; \Theta_1/\Psi$. By Proposition 5.4, there is a $\Sigma' \in \Omega_0$ with $\Theta_2/\Psi \; R_N \; \Sigma'$ and $\Sigma' \; R_G \; \Theta_1/\Psi'$. We get $\Sigma' \in \Theta_2 \cap \Psi'$, hence $\Theta_2 \in T_{\Psi'}$, and consequently, by $\Theta_2 \in T_\Psi$, $\Psi \sim_{\Theta_2} \Psi'$. Furthermore, by $\Sigma' = \Theta_2/\Psi'$ and $\Theta_2/\Psi' \; R_G \; \Theta_1/\Psi'$, we obtain $\Theta_2 <_{\Psi'} \Theta_1$. □

We summarize:

THEOREM 6.2. *For any irreflexive theory* Σ_0, *the quadruple* $\mathfrak{R}_0 = (T, W, <, \sim)$ *is a Kamp-frame.*

This frame is said to be **the canonical Kamp-frame for** Σ_0. By construction, $\mathfrak{R}_0$ is modally connected.

On the set of all constants we define a binary relation by

$$\begin{aligned} c \simeq d \quad &:\text{iff} \quad c \doteq d \in \Sigma, \text{ for at least one } \Sigma \in \Omega_0, \\ &:\text{iff} \quad c \doteq d \in \Sigma, \text{ for every } \Sigma \in \Omega_0. \end{aligned}$$

The equivalence can be obtained by Proposition 5.5 and the definition of Ω_0. $\simeq$ is an equivalence relation: this follows from Axiom (3) and Theorems (T-1) and (T-2). For a constant c, let $\bar{c}$ denote the equivalence class of c modulo $\simeq$. Given a m-ary relation symbol R, and constants $c_{i_1}, \ldots, c_{i_m}$ and $c_{j_1}, \ldots, c_{j_m}$ with $c_{i_k} \simeq c_{j_k}$, the following equivalence holds:

$$(+) \qquad R(c_{i_1}, \ldots, c_{i_m}) \in \Sigma \quad \text{iff} \quad R(c_{j_1}, \ldots, c_{j_m}) \in \Sigma$$

for every irreflexive theory $\Sigma \in \Omega_0$ (use Axiom (4)). Furthermore, given $\Sigma, \Sigma' \in \Theta$ and constants $c_i \simeq c_j$, we obtain from (NE) and Axiom (4):

$$(*) \qquad Ec_i \in \Sigma \quad \text{iff} \quad Ec_j \in \Sigma'.$$

On the canonical frame $\mathfrak{R}_0$ for Σ_0 we define:

$$\begin{aligned} U &:= \{\bar{c} \mid c \text{ is a constant}\}, \\ U_{\Theta,\Psi} &:= \{\bar{c} \in U \mid Ec \in \Theta/\Psi\}, \\ k(c) &:= \bar{c}, \\ p_{\Theta,\Psi}(R) &:= \left\{(\overline{c_{i_1}}, \ldots, \overline{c_{i_m}}) \mid R(c_{i_1}, \ldots, c_{i_m}) \in \Theta/\Psi\right\} \end{aligned}$$

for constants $c, c_{i_1}, \ldots, c_{i_m}$, m-ary relation symbols R, and $\Theta \in T_\Psi$. p is well defined: this follows from $(+)$. Let e denote that map which assigns to each Θ and Ψ with $\Theta \in T_\Psi$ the set $U_{\Theta,\Psi}$. e satisfies condition $(*)$ of Definition 3.1. Thus, the quintuple $\mathfrak{A}_0 = (\mathfrak{R}_0, U, e, k, p)$ is a Kamp-structure, **the canonical Kamp-structure for** Σ_0.

PROPOSITION 6.3. *For every sentence* σ *and each* $\Theta \in T_\Psi$, *the following equivalence holds:*

$$\mathfrak{A}_0 \models_{\Theta,\Psi} \sigma \quad \textit{iff} \quad \sigma \in \Theta/\Psi.$$

THEOREM 6.4 (Completeness of K^{irr}). *For every set* Σ *of* $\mathcal{L}$*-sentences which is consistent with respect to the calculus* K^{irr} *there is a Kamp-structure* $\mathfrak{A}$ *which satisfies* Σ.

In particular, each formula which is valid in all Kamp-structures is provable in the calculus K^{irr}.

Proof. Let Σ be a consistent set of sentences. By Corollary 4.5, there is an irreflexive theory Σ^* in a suitable extension $\mathcal{L}^*$ of the language $\mathcal{L}$ which includes Σ. Consider now the canonical Kamp-structure $\mathfrak{A}^*$ for Σ^* – note that this structure is defined with respect to irreflexive theories of the language $\mathcal{L}^*$. By Proposition 6.3, $\mathfrak{A}^*$ satisfies Σ^* and hence Σ at Θ_{Σ^*} and Ψ_{Σ^*}. But $\mathfrak{A}^*$ is a structure with respect to $\mathcal{L}^*$ and not with respect to $\mathcal{L}$. Hence, restrict the interpretation-functions k and p of $\mathfrak{A}^*$ to the language $\mathcal{L}$, and let $\mathfrak{A}$ denote the resulting Kamp-structure, which now is defined for the language $\mathcal{L}$. By an analogous argument as in the Coincidence-Theorem, it can be proved that $\mathfrak{A}$ and $\mathfrak{A}^*$ satisfy the same sentences of the language $\mathcal{L}$ at any moment in any world. Hence, $\mathfrak{A}$ satisfies Σ.

Let now φ be a formula which is not provable. Hence, the universal closure $\forall\varphi$ of φ is not provable, too. Consequently, the theory $\{\neg\forall\varphi\}$ is consistent, and hence satisfiable by a Kamp-structure. Thus, $\forall\varphi$ and φ are not valid in each Kamp-structure. □

7. CANONICAL TW-STRUCTURES AND COMPLETENESS

We redefine the relation $\approx$ of Section 6 as follows:

$$[\Sigma]_{\sim_G} \approx [\Sigma']_{\sim_G} \quad \text{:iff there are } \Gamma, \Gamma' \text{ with} \\ \Gamma \sim_G \Sigma, \Gamma' \sim_G \Sigma' \text{ and } \Gamma R_\Box \Gamma'.$$

Let Σ_0 be an arbitrary irreflexive theory, and let $\Omega_0, \Psi, \Psi_\Sigma, \Psi', \Psi_1, \ldots, \Theta, \Theta_\Sigma, \Theta', \Theta_1, \ldots$ be defined as in Section 6. Further, let $\Xi, \Xi_\Sigma, \Xi', \Xi_1, \ldots$ denote equivalence classes modulo $R_\Box$ which are included in Ω_0. We will show that any two equivalence classes Ψ and Ψ' are order-isomorphic with respect to the linear orders defined by R_G on Ψ and Ψ'. By Proposition 5.4(b) and $\Psi \approx \Psi'$, we obtain a map:

$$\iota_{\Psi,\Psi'} : \Psi \to \Psi',$$

that assigns to each theory $\Sigma \in \Psi$ the unique theory $\Sigma' \in \Psi'$ which is $R_\Box$-related to Σ.

PROPOSITION 7.1. *The map $\iota_{\Psi,\Psi'}$ is an order-isomorphism with respect to the linear orders defined by R_G on Ψ and Ψ'.*

Proof. The map $\iota := \iota_{\Psi,\Psi'}$ preserves the order because: Let Σ, Γ be in Ψ with $\Gamma R_G \Sigma$. Since Γ is irreflexive, there is a unary relation symbol R and a constant c with $\Box(Rc\wedge \mathrm{H}\neg Rc) \in \Gamma$. Because of $\Gamma R_\Box\, \iota(\Gamma)$ we obtain $Rc\wedge\mathrm{H}\neg Rc \in \iota(\Gamma)$. From $\mathrm{P}\Box(Rc\wedge\mathrm{H}\neg Rc) \in \Sigma$ we get $\Box\mathrm{P}(Rc\wedge\mathrm{H}\neg Rc) \in$

Σ, and hence $\mathrm{P}(Rc \wedge \mathrm{H}\neg Rc) \in \iota(\Sigma)$. By Theorem 4.7, there is a $\Gamma' \in \Psi'$ with $\Gamma' R_{\mathrm{G}}\, \iota(\Sigma)$ and $Rc \wedge \mathrm{H}\neg Rc \in \Gamma'$. By Lemma 5.3, we get $\Gamma' = \iota(\Gamma)$, and hence $\iota(\Gamma) R_{\mathrm{G}}\, \iota(\Sigma)$.

The map ι is bijective: consider the map $\iota' := \iota_{\Psi',\Psi}$ which assigns to each $\Sigma' \in \Psi'$ the unique $\Sigma \in \Psi$ which is $R_{\Box}$-related to Σ'. ι and ι' are inverse, i.e. for every $\Sigma \in \Psi$ and every $\Sigma' \in \Psi'$ we have:

$$\iota'\big(\iota(\Sigma)\big) = \Sigma \quad \text{and} \quad \iota\big(\iota'(\Sigma')\big) = \Sigma'.$$

This can be concluded as follows: Let $\Box(Rc \wedge \mathrm{H}\neg Rc)$ be in Σ. We get $\Box\Box(Rc \wedge \mathrm{H}\neg Rc) \in \Sigma$, consequently $\Box(Rc \wedge \mathrm{H}\neg Rc) \in \iota(\Sigma)$, and hence $Rc \wedge \mathrm{H}\neg Rc \in \iota'(\iota(\Sigma))$. By Lemma 5.3, Σ and $\iota'(\iota(\Sigma))$ are identical. The proof of the second identity is analogous. $\Box$

COROLLARY 7.2. *Let Ψ be an equivalence class modulo $\sim_{\mathrm{G}}$ and Ξ an equivalence class modulo $R_{\Box}$, both included by Ω_{Σ_0}. Then there exists exactly one irreflexive theory Σ such that*

$$\Sigma \in \Xi \cap \Psi.$$

Given equivalence classes Ξ and Ψ, let Ξ/Ψ denote the unique theory which lies in both Ξ and Ψ.

We can now give an alternative characterization of the equivalence classes modulo $\approx$: Each two equivalence classes modulo $\sim_{\mathrm{G}}$ are equivalent modulo $\approx$ if and only if there is an order-isomorphism between these equivalence classes such that this morphism assigns to each member of the first the unique $R_{\Box}$-related member of the second. In this case, the order-isomorphism is uniquely determined.

We are now going to define the canonical T×W-frame for Σ_0:

$$\begin{array}{rcl} T & := & \text{set of all equivalence classes } \Xi \text{ modulo } R_{\Box} \\ & & \text{which are included in } \Omega_0, \\ W & := & \text{set of all equivalence classes } \Psi \text{ modulo } \sim_{\mathrm{G}} \\ & & \text{which are included in } \Omega_0, \\ \Xi_1 < \Xi_2 & :\text{iff} & \Xi_1/\Psi \; R_{\mathrm{G}} \; \Xi_2/\Psi, \text{ for some (and hence} \\ & & \text{every) } \Psi \in W, \\ \Psi \sim_{\Xi} \Psi' & :\text{iff} & \Xi/\Psi \; R_{\mathrm{N}} \; \Xi/\Psi'. \end{array}$$

Note that by Corollary 7.2, we obtain a map

$$T \times W \to \Omega_0,$$

which assigns to each $\Xi \in T$ and each $\Psi \in W$ the unique irreflexive theory Ξ/Ψ which is in both equivalence classes Ξ and Ψ. This map is bijective, its inverse assigns to each irreflexive theory $\Sigma \in \Omega_0$ the pair $(\Xi_{\Sigma}, \Psi_{\Sigma})$.

PROPOSITION 7.3. *From* $\Psi \sim_{\Xi_1} \Psi'$ *and* $\Xi_2 < \Xi_1$ *it follows* $\Psi \sim_{\Xi_2} \Psi'$.

Let $\sim$ denote the map which assigns to each $\Xi \in T$ the relation $\sim_\Xi$ on W.

THEOREM 7.4. *Given an arbitrary irreflexive theory* Σ_0*, the quadruple* $\mathfrak{R}_0 := (T, <, W, \sim)$ *as defined above is a* T×W*-frame.*

This frame is said to be **the canonical T×W-frame for Σ_0**. Analogously to Section 6, we define the relation $\simeq$ on the set of all constants. Again, let $\bar{c}$ denote the equivalence class of c modulo $\simeq$. On the canonical TW-frame for Σ_0 we now define:

$$\begin{aligned} U &:= \{\bar{c} \mid c \text{ is a constant}\}, \\ U_{\Xi,\Psi} &:= \{\bar{c} \in U \mid Ec \in \Xi/\Psi\}, \\ k(c) &:= \bar{c}, \\ p_{\Xi,\Psi}(R) &:= \left\{(\overline{c_{i_1}}, \ldots, \overline{c_{i_m}}) \mid R(c_{i_1}, \ldots, c_{i_m}) \in \Xi/\Psi\right\} \end{aligned}$$

for constants $c, c_{i_1}, \ldots, c_{i_m}$, m-ary relation symbols R, and $\Xi \in T$, $\Psi \in W$. Let e denote the map that assigns to each $\Xi \in T$ and $\Psi \in W$ the set $U_{\Xi,\Psi}$. Obviously, the quadruple $\mathfrak{A}_0 = (\mathfrak{R}_0, U, e, k, p)$ is a T×W-structure, **the canonical T×W-structure for Σ_0**.

PROPOSITION 7.5. *For every sentence* σ *and each* $\Xi \in T$, $\Psi \in W$*, the following equivalence holds:*

$$\mathfrak{A}_0 \models_{\Xi,\Psi} \sigma \textit{ iff } \sigma \in \Xi/\Phi.$$

THEOREM 7.6 (Completeness of $K_\Box^{\text{irr}}$). *Every set of* $\mathcal{L}_\Box$*-sentences which is consistent with respect to the calculus* $K_\Box^{\text{irr}}$ *is satisfiable by a* T×W*-structure.*

In particular, each formula which is valid in all T×W*-structures is provable in the calculus* $K_\Box^{\text{irr}}$.

Proof. Cf. the proof of Theorem 6.4. □

8. SOME FINAL REMARKS

At first, we remark some general conclusions from the completeness results in Sections 6 and 7:

THEOREM 8.1 (Compactness). *A theory* Σ *is satisfiable* (*by a Kamp- resp.* T×W*-structures*) *if and only if each finite subset of* Σ *is satisfiable.*

Proof. Suppose that Σ is not satisfiable. Then, by the completeness theorems in Sections 6 and 7, Σ is inconsistent. By "syntactical compactness", we obtain a finite subset of Σ which is inconsistent, and hence, by the Soundness-Theorems 3.5, not satisfiable. The other direction is obvious. $\square$

THEOREM 8.2 (Löwenheim–Skolem). *Each satisfiable theory is satisfiable on a countable domain, i.e. given a satisfiable theory Σ, there is structure $\mathfrak{A}$ such that $\mathfrak{A}$ satisfies Σ and $|\mathfrak{A}|$ is countable.*

Proof. Each satisfiable theory is consistent, and by the construction of the domain in the canonical structure in question, each consistent theory is satisfiable on a countable domain: note that there are only denumerably many constants of the language $\mathcal{L}^*_{(\square)}$, and hence only countably many equivalence classes modulo $\simeq$. $\square$

We can strengthen the last result as follows:

THEOREM 8.1. *For each theory Γ which is satisfiable by a Kamp-structure there is a Kamp-structure $\mathfrak{A} = (\mathfrak{R}, U, \ldots)$ such that*:

(a) $\mathfrak{A}$ *satisfies* Γ,
(b) U *is countable,*
(c) $\mathfrak{R} = (T, W, <, \sim)$ *is modally connected, and*
(d) *for each $w \in W$, the set T_w is countable.*

Each theory which is satisfiable by a T×W-*structure is satisfied by a* T×W-*structure $\mathfrak{A} = ((T, \ldots), U, \ldots)$ with countable U and T.*

Proof. We may assume that Γ is irreflexive. Consider then the canonical Kamp-structure $\mathfrak{A}_0 = (\mathfrak{R}_0, \ldots)$ for Γ as defined in Section 6. We have to show that the canonical frame $\mathfrak{R}_0$ satisfies (d): Let $\Psi \in W$ be as in Section 6. For each $\Sigma \in \Psi$ there are a unique relation symbol R_{l_Σ} and a unique constant c_{k_Σ} with

(a) $\exists x \chi(R_{l_\Sigma}, x) \in \Sigma$ and there is no unary relation symbol with a lower index l such that $\exists x \chi(R_l, x) \in \Sigma$, and
(b) $\chi(R_{l_\Sigma}, c_{k_\Sigma}) \in \Sigma$ and there is no constant with a lower index k such that $\chi(R_{l_\Sigma}, c_k) \in \Sigma$.

Thus we have a map that assigns to each $\Sigma \in \Psi$ its "representative" pair $(R_{l_\Sigma}, c_{k_\Sigma})$. By Lemma 5.3, this map is injective. Obviously, the set of all representatives is countable, and hence Ψ is so, too. $\square$

Hence, by adding the usual axioms for rational flows of time to the calculus $K^{\text{irr}}_{\square}$, we obtain a sound and complete calculus with respect to $\mathbb{Q} \times W$-semantics.

Some remarks on the completeness proof in Section 7, and on the irreflexivity rule (R-Irr□) used therein may be in order. At first it should not go unmentioned that there are alternative rules that work as well: Replace $\chi(R, x)$ by one of the formulas $\Box Rx \wedge \mathrm{H}\Box\neg Rx$, $\Box(Rx \wedge \mathrm{G}\neg Rx)$ or $\Box(Rx \wedge \mathrm{H}\neg Rx \wedge \mathrm{G}\neg Rx)$.

Consider now the following rule

(**R**-Irr0) $\dfrac{\Box(Rc \wedge \mathrm{H}\neg Rc) \rightarrow \varphi}{\varphi}$ where R is a relation symbol, c an individual constant, and both do not occur in φ.

This rule is provable in the calculus $K_{\Box}^{\mathrm{irr}}$, and conversely (R-Irr□) is provable in the calculus $K_{\Box}$ + (R-Irr0). An interesting alternative to (R-Irr□) is to fix in (R-Irr0) an arbitrary unary relation symbol, for example E. Then the rule

(**R**-Irr1) $\dfrac{\Box(Ec \wedge \mathrm{H}\neg Ec) \rightarrow \varphi}{\varphi}$ where c is an individual constant that does not occur in φ

is equivalent to the rule

(**R**-Irr2) $\dfrac{\exists x\Box(Ex \wedge \mathrm{H}\neg Ex) \rightarrow \varphi}{\varphi}$

and this rule again is equivalent to the sentence

(**A**-Irr□) $\exists x\Box(Ex \wedge \mathrm{H}\neg Ex)$

regarded as an axiom. The calculus that is obtained in this manner can easily be shown to be adequate with respect to T×W-frames which satisfy this “irreflexivity axiom”.[10] For the completeness proof only maximal consistent theories with the ∃-property have to be considered instead of irreflexive ones. Hence the constructions in Section 4 become quite comfortable and similar to those in ordinary modal logic. But (A-Irr□) is not plausible as a special existence postulate for objects in the normal sense. (An example for an unusual interpretation of the language $\mathcal{L}_{\Box}$ is presented below.)

With respect to Kamp-semantics, by the axiom

(**A**-IrrN) $\exists x\mathrm{N}(Ex \wedge \mathrm{H}\neg Ex)$

we can prove completeness analogously. Moreover, by Axiom (NE) and Theorem (T-HN), we obtain that the axiom

(**A**-Irr) $\exists x(Ex \wedge \mathrm{H}\neg Ex)$

works as well. But even if one is inclined to consider (A-Irr) to be true (true in the real world) it is questionable whether (A-Irr) expresses a logical truth.

Hence, the question arises whether there is a certain predicate $\varphi(x)^{11}$ for which the formula

$$(*) \qquad \exists x \Box(\varphi(x) \wedge \mathrm{H}\neg\varphi(x))$$

can be regarded as an axiom. Note at first that by the Axiom of Choice, for each T×W-structure $\mathfrak{A} = ((T, \ldots), U \ldots)$ in which a formula of the form $(*)$ is valid there is an injective map from T into U. For the converse assume that $\mathfrak{A} = ((T, \ldots), U, \ldots)$ is a T×W-structure such that there is an injective map ι from T into U. Introduce a new unary relation symbol R^* and define a T×W-structure $\mathfrak{A}^*$ with respect to the language $\mathcal{L}_\Box \cup \{R^*\}$ such that (a) $\mathfrak{A}^*$ aggrees with $\mathfrak{A}$ in the interpretation of all symbols of the language $\mathcal{L}_\Box$, and (b) the interpretation of R^* is defined by:

$$p^*_{t,w}(R^*) := \bigcup_{t' \leqslant t} \{\iota(t')\}.$$

Obviously, in $\mathfrak{A}^*$ and $\mathfrak{A}$ the same formulas of the language $\mathcal{L}_\Box$ are valid, and $\mathfrak{A}^*$ validates the sentence $\exists x \Box(R^*x \wedge \mathrm{H}\neg R^*x)$. Thus, the answer to the question above is: The plausibility of any irreflexivity axiom of the form $(*)$ depends on the question whether time-points are uniquely representable in the universe of discourse under consideration.

But all these observations do not elucidate the role of irreflexivity rules. Now, in view of the completeness proof in Section 7, sentences of the form $\chi(R, c)$ can be regarded as a kind of *dates*, as terms that denote time-points of momentary states: With respect to canonical T×W-frames, momentary states (or: moments) can be identified with irreflexive theories. Simultaneous ($R_\Box$-related) moments have the same date, nonsimultaneous moments do not (cf. Proposition 5.4). Each irreflexive theory contains a sentence of the form $\chi(R, c)$, hence every irreflexive theory contains a sentence which expresses its date. By its date, an irreflexive theory is uniquely determined in its history (cf. Lemma 5.3). Indeed, what is said by the irreflexivity rule is: each momentary state has a date, and by its date a momentary state can uniquely be denoted in its history; or alternatively, each momentary state contains a date proposition by which the position in its history is uniquely characterized. This expresses the irreflexivity of time.

This interpretation provides an example where (A-Irr$\Box$) can be assumed to be analytically true. Suppose that we are interested in a theory of propositions in the framework of T×W-semantics. Let U denote a set of possible (not necessarily true or eternal) propositions. Our first attempt to fix the interpretation of E is:

> Ex is true iff the proposition denoted by x is a proposition which holds (is true) at the momentary state (of the actual world); i.e. Ex is true at t and w iff the proposition denoted by x holds at t and w.

But with respect to this interpretation, Axiom (NE) is not plausible. Not every proposition which holds at some t and w does also hold in each world which shares up to (inclusively) t the same history with w. As an example consider a proposition about a future state of the world.[12]

Our second attempt to fix the interpretation of E is:

> Ex is true iff the proposition denoted by x is a (true) proposition *about* the present momentary state (of the actual world); i.e. Ex is true at t and w iff the proposition denoted by x holds at t and w, *independently* of what happened in w before t and of what will happen in w afterwards.

The second reading implies the first, but not vice versa. Then PEa (resp. FEa) may be read as: "a is a proposition about some past (future) momentary state (of the actual world)".[13] With respect to this interpretation, Axiom (NE) becomes plausible: If the proposition denoted by x is a proposition about the present momentary state, then this proposition is in each world which shares up to now the same history with the real world a proposition about the present momentary state (in that world).

With respect to both interpretations, Axiom (A-Irr$\Box$) is plausible: for every time-point t, there is a proposition which holds in any world at this time-point (resp.: which is a proposition about the present momentary state in any world), but never before (and never afterwards), namely the proposition expressed by "Now it is Dat_t" where Dat_t is a name for t.

NOTES

[1] Cf. Gabbay et al. (1994), Lemma 6.7.2, p. 212.

[2] Cf. Hughes & Cresswell (1996), pp. 303ff.

[3] Cf. Hughes & Cresswell (1996), pp. 327–328.

[4] Note that the calculi K and K^{irr} are defined with respect to the languages $\mathcal{L}$ and $\mathcal{L}_{\Box}$, while $K_{\Box}$ and $K^{\mathrm{irr}}_{\Box}$ are only defined with respect to $\mathcal{L}_{\Box}$.

[5] There may be interpretations of E where condition $(*)$ and the corresponding axiom (NE) are not plausible, cf. Section 8 of the paper.

[6] This means: A T×W-frame $\mathfrak{R}$ which is infinite in the past is modally connected if and only if formula (2) is valid in each T×W-structure on $\mathfrak{R}$. This does not imply that (2)

axiomatizes modal connectedness. As far as I can see, with respect to T×W-frames modal connectedness is axiomatizable by each of the formulas:

$$(2\delta)\ \Box\delta \wedge \Diamond\psi \rightarrow \mathrm{M}(\delta \wedge \psi) \vee \mathrm{PMF}(\delta \wedge \psi),$$

where δ is one of the formulas $\varphi \wedge \mathrm{G}\neg\varphi$, $\varphi \wedge \mathrm{H}\neg\varphi$ or $\varphi \wedge \mathrm{H}\neg\varphi \wedge \mathrm{G}\neg\varphi$. If we assume infinity in the past, (2δ) can be reduced to

$$(2^*\delta)\ \Box\delta \wedge \Diamond\psi \rightarrow \mathrm{PMF}(\delta \wedge \psi).$$

[7] In the literature some similar concepts are discussed: "theory that contains witnesses" (Chang, Keisler (1973)), "theory which has the ∀-property" (Hughes, Cresswell (1984)) and "ω-saturated theory" (Gabbay et al. (1994)). With respect to maximal consistent theories, all these concepts coincide.

[8] Note that this condition is expressed more exactly as follows: "If σ is a modaltypical sentence in Σ and $\Phi(\varphi_n) := \ldots$ is a decomposition of σ, then there is ...." As an example, consider a sentence of the form $\mathrm{F}(\varphi \wedge \mathrm{M}\psi)$. Now, if Σ satisfies the (Irr)-property and contains $\Phi(\psi) := F(\varphi \wedge \mathrm{M}\psi)$, then there is

(1) a unary relation symbol R with $\Phi(\psi \wedge \exists x\chi(R, x)) = \mathrm{F}(\varphi \wedge \mathrm{M}(\psi \wedge \exists x\chi(R, x))) \in \Sigma$;

and, if we apply the condition above to the decomposition $\Phi'(\varphi \wedge \mathrm{M}\psi) := \mathrm{F}(\varphi \wedge \mathrm{M}\psi)$, there is

(2) a unary relation symbol R' with $\Phi'(\varphi \wedge \mathrm{M}\psi \wedge \exists x\chi(R', x)) = \mathrm{F}(\varphi \wedge \mathrm{M}\psi \wedge \exists x\chi(R', x)) \in \Sigma$.

[9] Given a formula ψ, let $\psi(S/R)$ be the formula which results by replacing each occurrence of S in ψ by R – provided that R and S have the same number of places.

[10] This axiom corresponds to the irreflexivity axiom $\exists p\Box(p \wedge \mathrm{H}\neg p)$ for languages with propositional quantification (cf. Gabbay et al. (1994), pp. 311ff).

[11] We write $\varphi(x)$ in case that x is the only variable which occurs free in φ.

[12] For example: assume that in the actual world, I have begun climbing some peak, and that I will succeed in climbing that peak; in some other world which shares up to now the same history with the actual world, I have begun climbing as well, but in that world I will break off and turn back. Hence, from the fact that I will successfully climb that peak, it does not follow that this is necessarily the case.

[13] Note that not each proposition which is expressed by a statement in present (past/future) tense is a proposition about a present (past/future) momentary state. Cf. Meixner (1987), pp. 111ff.

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