Branching Allen

Reasoning with Intervals in Branching Time

Marco Ragni and Stefan Wölfl

Institut für Informatik, Albert-Ludwigs-Universität Freiburg,
Georges-Köhler-Allee, 79110 Freiburg, Germany
{ragni, woelfl}@informatik.uni-freiburg.de

Abstract. Allen's interval calculus is one of the most prominent formalisms in the domain of qualitative spatial and temporal reasoning. Applications of this calculus, however, are restricted to domains that deal with linear flows of time. But how the fundamental ideas of Allen's calculus can be extended to other, weaker structures than linear orders has gained only little attention in the literature. In this paper we will investigate intervals in branching flows of time, which are of special interest for temporal reasoning, since they allow for representing indeterministic aspects of systems, scenarios, planning tasks, etc. As well, branching time models, i. e., treelike non-linear structures, do have interesting applications in the field of spatial reasoning, for example, for modeling traffic networks. In a first step we discuss interval relations for branching time, thereby comprising various sources from the literature. Then, in a second step, we present some new complexity results concerning constraint satisfaction problems of interval relations in branching time.

1 Introduction

Allen's interval calculus is one of the most prominent formalisms in the domain of qualitative spatial and temporal reasoning. But applications of this calculus are restricted to domains in which intervals in *linear* flows of time are considered. Surprisingly, the question of how the ideas of Allen's calculus can be extended to other, weaker structures than linear orders has gained only little attention in the literature. In this paper we will focus on intervals in branching time. The basic idea of branching time is that at each moment there exists only one possible past, but many possible futures. Hence branching flows of time, which can be modeled by tree-like structures are of special interest for temporal reasoning since they allow for representing indeterministic aspects of systems, scenarios, and planning tasks.

In modal logic, branching time models have been studied intensely in the past two decades. Originating from philosophical logic (cf. [29]), where branching time logics have been investigated for analyses of indeterminism, causality, and action-theoretical concepts, branching time logics such as CTL and CTL* (cf. [10]) and their multi-agent extensions ATL and ATL* (cf. [3]) have been discussed as specification languages, mainly for model checking purposes of closed, reactive systems as well as of systems that interact with their environment.

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Allen's interval algebra is a reasoning formalism in the spirit of Hayes' naïve manifesto [14]. If the instants of a linear flow of time count as primary entities (which is, for example, the point of view of the so-called point algebra of linear time), intervals are *sets* of instants. Thus, the interval algebra can be seen as a shift of perspective from first-order entities (instants) to second-order entities (intervals). This paper will deal with the analogous change of perspective, from moments in branching time towards intervals in branching time.

Algebraic aspects of the point algebra of branching time were first investigated by Düntsch, Wang, and McCloskey [9]. Hirsch [17] showed that local consistency is insufficient for satisfiability testing for the point algebra of branching time. Contrary to the point algebra of linear time, satisfiability testing for branching time is NP-hard. Broxvall [6] discussed tractable subclasses of the point algebra of branching time. Tractable subclasses of the interval algebra of linear time were identified by Nebel and Bürckert [26] and by Ligozat [24].

What is the motivation for considering branching time, tree-like structures? First, tree-like structures are a natural choice for modeling temporal aspects of events. For example, Kutschera [18] defined events as sets of closed intervals in branching time. Tree-like structures are used to model the various courses the world might take. A (complete) branch of a tree represents one specific way in which the world can evolve. The basic idea, then, is to identify an event with the set of its occurrences in time, i. e., with the set of its temporal extensions. An event can occur in many branches — an event is said to occur in a branch if one of its instances is completely contained in that branch. But since events are understood as singular events, an event can occur only once in a branch. The main requirement of Kutschera events is that when an event occurs in two branches that overlap while the event occurs in at least one of them, then the event starts in both branches at the same moment. This little discussion already indicates how reasoning with Allen-style interval relations (adapted for branching flows of time) could be used for reasoning about events.

With respect to more spatial domains, a theory of intervals in tree-like structures may have interesting applications, for example, when routes in traffic networks are represented by qualitative concepts. Of course, most street networks are not tree-like, but many railroad networks are. Modeling street networks by tree-like structures may be applicable, especially when one focuses on "small" traffic scenarios. To illustrate this, let us assume that the spatial configuration of an intersection of highways is to be represented by qualitative means. Then one can distinguish lane segments according to the traffic regulations that hold within these segments. These segments may be related to each other by any of the base relations of Allen's interval algebra. For example, a segment in which passing is forbidden may start a segment in which speed is limited. Thus, lane segments are a natural candidate for intervals. But also cars and accumulations of cars can be represented by intervals. Hence, a congestion in a lane segment can be modeled by two intervals, with one contained in the other. The branching aspect comes into play since, in this qualitative language, we can describe a car driving off the road or a road connecting two highways.

Finally, branching structures can also be applied in planning domains. Planning deals with the question of how a certain goal state can be reached from an initial state by executing a sequence of actions. Usually, planning tasks can be modeled by Kripke style transition graphs, and these graphs can be *unwinded* to tree-like structure. The method of unwinding a transition graph is applied implicitly, when heuristic forward search is used in planning algorithms.

The paper is organized as follows: In section 2 we review some basic concepts of the theory of tree-like structures, and we sketch some results concerning the point algebra of branching time. In section 3 we present the base relations between intervals in branching time. More precisely, we define two algebras of base relations, where one is a refinement of the other one. Section 4 deals with the conceptual neighborhood graph for interval relations in branching time and discusses its relationship to the linear time version. In section 5 we investigate the computational complexity of constraint satisfaction problems of the algebras presented previously. In particular, we show that the satisfiability problem with respect to the coarser algebra of interval relations is NP-complete. In section 6 we work out some of the particularities of the composition table of interval relations in branching time. Finally, section 7 summarizes the results of the paper and gives a short overview of some questions that are left open in this paper.

2 Branching Time Theory

To begin with, let us recall some basic concepts from the theory of tree-like structures.

Definition 1. A *tree* is an ordered pair $\mathcal{B} = \langle T, \prec \rangle$ consisting of a non-void set of *nodes*, T, and a binary relation, $\prec$, satisfying the following properties:

(a) $\prec$ is a partial order on T, i. e., $\prec$ is irreflexive and transitive;
(b) $\prec$ does not allow for *backward branching*, i. e., $\prec$ is linear-to-the-left;[1]
(c) T is *connected* via $\prec$, i. e., for all $t, t' \in T$ with $t \not\prec t'$, $t' \not\prec t$, and $t \neq t'$, there exists a $t'' \in T$ such that $t'' \prec t$ and $t'' \prec t'$.

We read $t \prec t'$ as “t is earlier than t'”. Symbols such as $\preceq$, $\succ$, and $\succ$ are used in the natural manner. For sets X and X' of nodes, let $X \preceq X'$ ($X \prec X'$) be defined as: for all $t \in X$ and $t' \in X'$, it holds that $t \preceq t'$ ($t \prec t'$). Finally, $X \preceq t$ is an abbreviation of $X \preceq \{t\}$, etc. Nodes t and t' are said to be *unrelated*, $t \parallel t'$, if neither $t \preceq t'$ nor $t' \preceq t$.

A *chain* of nodes is a set of nodes that is linearly ordered by the relation of earlier-than, i. e., each pair of nodes chosen in the chain are comparable with respect to $\prec$. A chain k is said to be *upper-bounded* if there is a node t with $k \preceq t$. In an analogous manner concepts such as *lower-bounded* or *bounded* are introduced.

Maximal $\prec$-chains in a tree $\mathcal{B}$ are said to be *branches*, and the set of all branches is denoted by B. For a given node t, let $B\langle t \rangle$ be the set of all branches that contain t as an element. Furthermore, we will use the following terminology: Branches b and b' are *undivided* at node t if there exists a node $t' \succ t$ with $t' \in b \cap b'$. Branches b and b' *split* at node t if t is the maximal element of $b \cap b'$. Note that the intersection of a pair of

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[1] This means that for all nodes t and t', it holds either $t \prec t'$ or $t = t'$ or $t' \prec t$, provided there is an t'' with $t, t' \prec t''$.

branches need not have a maximal element, even if they intersect. A node t is a *splitting point* if there exist branches that split at t. Branches b and b' are *separated* at node t if either $t \in b \setminus b'$ or $t \in b' \setminus b$.

With these notations we can replace condition 1 (c) by each of the following conditions:

(c_1) If $t \parallel t'$, then there exists a t'' with $t'' \prec t$ and $t'' \prec t'$.
(c_2) There exists a t'' with $t'' \preceq t$ and $t'' \preceq t'$.
(c_3) For each pair of branches b and b', $b \cap b' \neq \emptyset$.

And if $\prec$ is infinite-to-the-left, then condition (c) is equivalent to:

(c_4) There exists a t'' with $t'' \prec t$ and $t'' \prec t'$.

A tree $\mathcal{B} = \langle T, \prec \rangle$ is said to be *dense* if for each pair of nodes $t, t' \in T$ with $t \prec t'$, there exists a node $t'' \in T$ such that $t \prec t'' \prec t'$; $\mathcal{B}$ is said to be *branching dense* if, for each pair of nodes $t, t' \in T$ with $t \prec t'$, there exists a $t'' \in T$ such that $t \prec t''$ and $t' \parallel t''$. Obviously, density does not follow from branching density, and vice versa. Note that there exist finite and branching dense trees, but that no tree is both finite and dense. Nevertheless, branching density is a very strong condition since, in a finite branching dense tree, each node that is not the endpoint of some branch is a splitting point. Finally, it is worth mentioning that trees are not required to have roots.

The intended models for Allen's calculus are dense linear flows of times without endpoints, for example, the linear order of the rationals or that of the reals. A typical example of a dense and infinite tree is any instance of a $\mathbb{Q}$- or an $\mathbb{R}$-tree.

Definition 2. A tree $\mathcal{B} = \langle T, \prec \rangle$ is said to be a $\mathbb{Q}$*-tree* (an $\mathbb{R}$*-tree*, resp.) if there exists a family $(\iota_b)_{b \in B}$ of order isomorphisms $\iota_b \colon b \longrightarrow \mathbb{Q}$ (or $\iota_b \colon b \longrightarrow \mathbb{R}$, resp.) such that for all $b, b' \in B$ and each $x \in b \cap b'$, $\iota_b(x) = \iota_{b'}(x)$.

Hence in a $\mathbb{Q}$-tree, each node of a branch can be labeled by a rational number via an order isomorphism, and the labeling of nodes in one branch respects the labeling of nodes in another branch as long as both branches intersect.

In the class of all dense trees, it is reasonable to distinguish two tree types with regard to the structure of how branches of the tree actually split:

(a) $\mathcal{B}$ is said to be of *type 1* if, for each pair of distinct branches $b, b' \in B$, the intersection of b and b' has a maximum, i. e., there exists a node at which b and b' split.[2]
(b) $\mathcal{B}$ is said to be of *type 2* if, for each pair of distinct branches b and b', the intersection of b and b' has no maximum, i. e., b and b' are undivided at each node $t \in b \cap b'$.

This list is not exhaustive since further splitting types can be defined. As well, there are trees that do not have a uniform splitting type. However, when a scenario is represented in terms of $\mathbb{Q}$-trees, it is reasonable to fix a specific splitting structure according to the typology presented here. Note that finite trees are always of type 1. We will discuss this point in more detail later.

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[2] This condition is also known as the *semi-lattice condition*.

Table 1. The composition table of the point algebra for branching flows of time (cf. [6])

	$\prec$	$\succ$	$\parallel$	$=$
$\prec$	$\prec$	$\prec,=,\succ$	$\prec,\parallel$	$\prec$
$\succ$	$\top$	$\succ$	$\parallel$	$\succ$
$\parallel$	$\parallel$	$\succ,\parallel$	$\top$	$\parallel$
$=$	$\prec$	$\succ$	$\parallel$	$=$

In Allen's interval calculus, an interval is represented by a pair of points, namely the start and the endpoint of the interval. Thus in a weak sense, reasoning with intervals can be reduced to reasoning with points of the underlying linear flow of time. On the other hand, reasoning with instants of a linear flow of time can be done by employing the so-called point algebra for linear time, PA_{lin}. The point algebra for branching flows of time, PA_{br}, has been investigated by Broxvall and Jonsson [7, 6]. In PA_{br}, the relations $\prec$, $=$, $\succ$, and $\parallel$ count as base relations. More precisely, these relations are the atoms of the relation algebra that is defined on the set of all (set-theoretical) unions of base relations via the composition table shown in Table 1. In qualitative reasoning, unions of base relations are considered to model imprecise knowledge in a given scenario.

Note that given an atomic relation algebra A with finite atom set $B(A)$ (i.e., $B(A)$ is the set of all base relations), each relation $r \in A$ can be written in a unique manner as a union of base relations $b_1, \ldots, b_n$. Hence, algebraic functions such as composition, converse, intersection, union, and complement, can be computed from base relations by applying the following equations:

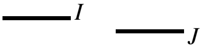
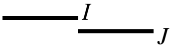
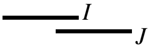
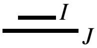
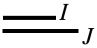
$$\begin{aligned}
(b_1 \cup \cdots \cup b_n) \circ (b'_1 \cup \cdots \cup b'_m) &= \bigcup_{1 \le i \le n, 1 \le j \le m} (b_i \circ b'_j) \\
(b_1 \cup \cdots \cup b_n)^{-1} &= (b_1^{-1} \cup \cdots \cup b_n^{-1}) \\
(r \cap r') &= \bigcup \{ b \in B(A) : b \subseteq r \text{ and } b \subseteq r' \} \\
(r \cup r') &= \bigcup \{ b \in B(A) : b \subseteq r \text{ or } b \subseteq r' \}
\end{aligned}$$

It is worthwhile to remark that the general constraint satisfaction problem for PA_{br} is NP-hard, while it is in P for the point algebra of linear time. Broxvall [6] identified five maximal tractable subsets of the point algebra for branching time and showed that these are the only maximal tractable subsets.

3 Intervals and Branching Time

As said before, in Allen's theory intervals are identified with pairs of points of a given linear order $\langle T, < \rangle$, which is assumed to be isomorphic to the linear order of the rationals. More precisely, an *Allen interval* is a pair of points $\langle t_1, t_2 \rangle \in T^2$ with $t_1 < t_2$. By considering start and endpoints of intervals, Allen identified thirteen jointly exhaustive and pairwise disjoint relations known in the literature as the Allen 13 relations (cf. Table 2).

Table 2. The 13 base relations of Allen's interval algebra

Relation	Converse	Pictorial Representation
$I\,b\,J$	$J\,bi\,I$	
$I\,m\,J$	$J\,mi\,I$	
$I\,o\,J$	$J\,oi\,I$	
$I\,d\,J$	$J\,di\,I$	
$I\,s\,J$	$J\,si\,I$	
$I\,f\,J$	$J\,fi\,I$	
$I\,e\,J$	$J\,e\,I$	

In contrast to Allen's theory, we will use the notion of *interval* in the following sense:

Definition 3. Let $\mathcal{B} = \langle T, \prec \rangle$ be a tree. An *interval* in $\mathcal{B}$ is a convex and bounded $\prec$-chain in T.

It is worthwhile to recall that in $\mathbb{R}$ each interval (in terms of this definition) can be represented as an Allen interval, i. e., via start and endpoint. With respect to the linear order of the rationals, this situation is different since $\mathbb{Q}$ is not Dedekind-complete. Note that representing intervals as Allen intervals is not unique. For example, the intervals $(0,1)$, $(0,1]$, $[0,1)$, and $[0,1]$ do have the same Allen representation $\langle 0,1 \rangle$. In Allen's approach this in some sense imprecise representation is chosen intentionally to abstract from the exact boundary structure of intervals.[3] But when we are to present models for Allen relations it is reasonable, for the sake of simplicity, to fix the exact structure of intervals because the boundary structure depends on the interpretation of Allen primitives such as "starts", "meets", etc. For if the primitive "meet" is understood in the sense that "I meets J" entails that I and J intersect, then Allen intervals are understood as closed intervals, i. e., as intervals of the form $[t,t']$. If "I meets J" means "there is no interval K with $I \leq K \leq J'$", then we are free to fix the boundary structure.

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[3] Note that there are two distinct meanings in which abstraction from the boundary structure can be understood. Let us assume that we are given a set of Allen relation constraints, C, a linear flow of time, and an assignment of the variables occurring in C, a, that satisfies C and interpretes intervals as, say, closed intervals. In its first sense, then, abstraction means that we obtain a model of C by replacing a by any assignment a' that interpretes *some* of the intervals in C as open intervals. In the weaker sense, abstraction means that we have to replace *uniformly*, for example, a closed interval interpretation by an open interval interpretation to obtain a satisfying model. In what is to follow we will use abstraction in the first sense.

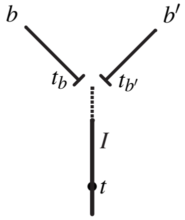


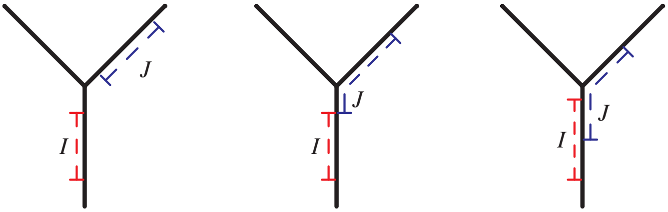
Fig. 1. A branching scenario of type 2. Interval I starts at t and continues until b and b' are split. I does not contain one of its supreme points, t_b and $t_{b'}$. How could I be represented as an Allen interval?

In branching time the boundary structure of intervals is more crucial than it is in linear flows of time. To illustrate this, let us assume that $\mathcal{B} = \langle T, \prec \rangle$ is an $\mathbb{R}$-tree of type 2. Let b and b' be branches that intersect, and choose $t \in b \cap b'$. Consider then the interval $I := \{t' \in b \cap b' : t \preceq t'\}$. Obviously, in each of the branches b and b' there exists a supremum of I, t_b and $t_{b'}$ respectively, but t_b is not contained in b', and, vice versa, $t_{b'}$ is not contained in b. Now if we represented I as the Allen interval $\langle t, t_b \rangle$, then the interval $I = [t, t_b) = [t, t_{b'})$ will have the same Allen representation as the interval $I_b := [t, t_b]$. But from a conceptual point of view, I and I_b are essentially distinct, since I is contained in b', while I_b is not. The difference between I and I_b can even be expressed in terms of Allen relations. For if t' is a node with $t' \succ t_{b'}$, then it makes perfect sense to say that interval I meets interval $J := [t_{b'}, t')$, while one would be inclined to say that I_b does not meet J.

This small discussion shows that the Allen representation of intervals is at least problematic or even inappropriate in the context of branching time. One could argue that the example just discussed crucially depends on the splitting structure of the tree. But analogous examples could be constructed if the tree at hand is of type 1.

Following we will present in an informal manner a set of 24 relations that may hold between two intervals in a tree. To our knowledge, these relations were first discussed by Anger, Ladkin, and Rodriguez [4], and we widely adopt the notations from their paper. To present the set of 24 base relations, we will assume that intervals are closed, in the sense that they contain a minimal and a maximal element (a start and an endpoint): this is advantageous, since in this case we need not take care of the splitting type of the tree. Furthermore, we will use concepts such as "I is connected with J" or "I meets J" in the sense that intervals I and J do intersect. Let now I and J be intervals. We start by presenting those relations where both intervals I and J are contained in some branch b. Obviously, these relations are exactly the 13 base relations known from Allen's interval algebra, now in a branching context (cf. Fig. 2). In what follows these relations will be referred to as *linear relations*.

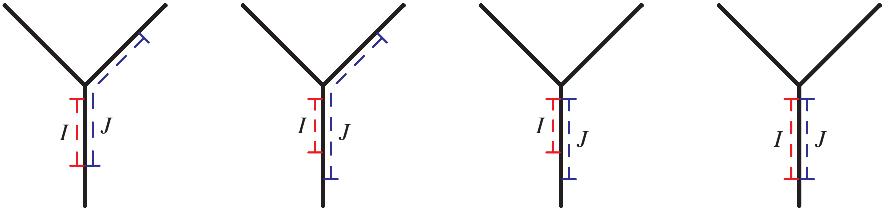
Let us now assume that I is completely contained in some branch b, while J is not. From this it follows that there is a subinterval of I that is not comparable to J via one of the Allen relations. Then we may distinguish two cases: In the first case another subinterval, more precisely an initial segment, of I is related to J by one of the Allen relations. This enables us to differentiate seven additional relations (cf. Fig. 3).



(a) *I* before *J* (symb.: *I b J*, converse: *J bi I*)

(b) *I* meets *J* (symb.: *I m J*, conv.: *J mi I*)

(c) *I* overlaps *J* (symb.: *I o J*, conv.: *J oi I*)



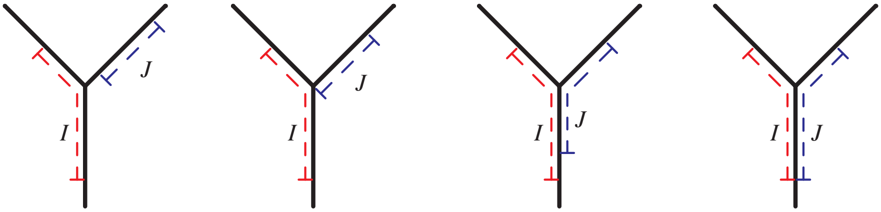
(d) *I* starts *J* (symb.: *I s J*, conv.: *J si I*)

(e) *I* during *J* (symb.: *I d J*, conv.: *J di I*)

(f) *I* finishes *J* (symb.: *I f J*, conv.: *J fi I*)

(g) *I* equals *J* (symb.: *I e J*, symmetric)

Fig. 2. The 13 Allen relations in a branching context



(a) *I* partially before *J* (symb.: *I pb J*, conv.: *J pbi I*)

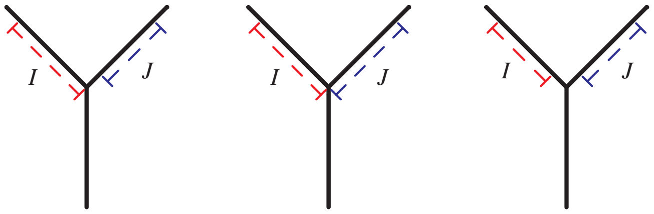
(b) *I* partially meets *J* (symb.: *I pm J*, conv.: *J pmi I*)

(c) *I* partially overlaps *J* (symb.: *I po J*, conv.: *J poi I*)

(d) *I* partially starts *J* (symb.: *I ps J*, symmetric)

Fig. 3. Splitting point within one of the intervals. Seven relations can be distinguished if there exist two subintervals of *I* (or *J*, resp.), where one is comparable to *J* (or *I*, resp.) via Allen relations, and the other one is not

In the second case, we may assume that no initial segment of *I* is related to *J* via some Allen relation. In this case we can distinguish four further relations as depicted in Fig. 4. Note that these relations can be characterized by quantifying over intervals. For example, if we assume that *I* and *J* are not comparable by one of the 20 relations presented before, then *I* is unrelated to *J* if and only if there are intervals I' and J' such



(a) I adjacent to J (symb.: $I\ a\ J$, conv.: $J\ ai\ I$)

(b) I touches J (symb.: $I\ t\ J$, symmetric)

(c) I unrelated to J (symb.: $I\ u\ J$, symmetric)

Fig. 4. Splitting point preceding the intervals. Four relations can be distinguished if there is no subinterval of I (or J, resp.) that is comparable to J (or I, resp.) by an Allen relation

that I' meets I, J' meets J, and I' and J' are not related by one of the 13 Allen relations. Then the relations a and t can be defined as follows: I is adjacent to J if each interval that finishes I is unrelated to J and if no interval that starts I is before J (note that this definition depends on the assumption that the tree at hand is dense). And, I touches J if for each pair of intervals I' and J' with $I'\ f\ I$ and $J'\ f\ J$, I' is unrelated to J'.

It is worthwhile to remark that only 19 base relations are definable if we restrict consideration to the start and endpoints of intervals. "Definable" here means definable in the constraint language of endpoints, which does not allow quantification over splitting points. Table 3 provides an overview (cf. Euzenat [11]). By comparing the defining constraints for the set of 19 base relations (depicted in Table 3) with those constraints for the refined set of 24 base relations (cf. figures 2–4 and Table 4), one can readily check that the following equivalences hold:

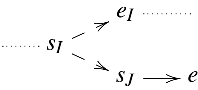
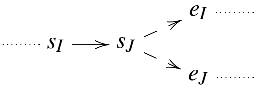
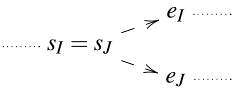
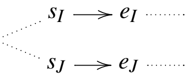
$$\begin{aligned} I\ ib\ J &\iff I\ pb\ J \vee I\ a\ J \\ I\ ie\ J &\iff I\ ps\ J \vee I\ t\ J \\ I\ im\ J &\iff I\ pm\ J \vee I\ po\ J \end{aligned}$$

The following remarks will round off the presentation of interval relations in branching time.

Remark 4. If connectedness of trees (condition 1(c)) is not enforced, a tree may have a forest-like structure consisting of several (genuine) tree components. Then the relation u can be partitioned into two subrelations, namely "unrelated, but contained in the same tree component" and "contained in disjoint tree components". Here the term "tree component" is used to denote a tree in which all branches are connected.

Remark 5. The 19 base relations as well as the 24 base relations defined in this section can be shown to be jointly exhaustive and pairwise disjoint. This follows immediately from the defining constraints presented in tables 3 and 4. But this result depends crucially on the closed-interval interpretation we have chosen in the presentation. A more general result can be obtained by introducing a first order theory of intervals in branching time in the style of Allen and Hayes' axiomatization of the meet relation for linear time (cf. [1, 15]). Yet, a detailed discussion of such a theory would go beyond the scope of this paper.

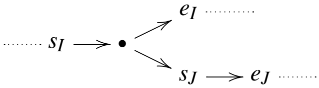
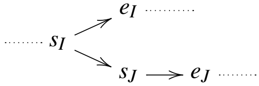
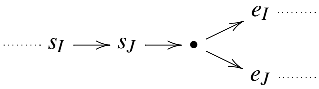
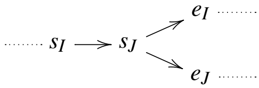
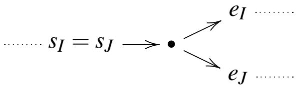
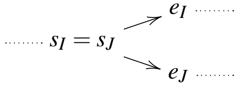
Table 3. A set of 19 base relations definable by interval endpoints in branching time

Symbol	Relation	Defining Constraints	Pictorial Representation
b (conv.: bi)	I before J	$e_I \prec s_J$	$\cdots\, s_I \longrightarrow e_I \longrightarrow s_J \longrightarrow e_J \,\cdots$
m (mi)	I meets J	$e_I = s_J$	$\cdots\, s_I \longrightarrow e_I = s_J \longrightarrow e_J \,\cdots$
o (oi)	I overlaps J	$s_I \prec s_J, s_J \prec e_I,$ $e_I \prec e_J$	$\cdots\, s_I \longrightarrow s_J \longrightarrow e_I \longrightarrow e_J \,\cdots$
d (di)	I during J	$s_J \prec s_I, e_I \prec e_J$	$\cdots\, s_J \longrightarrow s_I \longrightarrow e_I \longrightarrow e_J \,\cdots$
s (si)	I starts J	$s_I = s_J, e_I \prec e_J$	$\cdots\, s_I = s_J \longrightarrow e_I \longrightarrow e_J \,\cdots$
f (fi)	I finishes J	$s_J \prec s_I, s_I \prec e_J,$ $e_J = e_I$	$\cdots\, s_J \longrightarrow s_I \longrightarrow e_J = e_I \,\cdots$
e	I equals J	$s_I = s_J, e_I = e_J$	$\cdots\, s_I = s_J \longrightarrow e_I = e_J \,\cdots$
ib (ibi)	I initially before J	$s_I \prec s_J, s_J \parallel e_I$	
im (imi)	I initially meets J	$s_I \prec s_J, s_J \prec e_I,$ $e_I \parallel e_J$	
ie	I initially equals J	$s_I = s_J, e_I \parallel e_J$	
u	I unrelated to J	$s_I \parallel s_J$	

This table shows all possibilities of how two Allen intervals $I = \langle s_I, e_I \rangle$ and $J = \langle s_J, e_J \rangle$ can be related in a tree $\mathcal{B} = \langle T, \prec \rangle$. It is always assumed that $s_I \prec e_I$ and $s_J \prec e_J$.

Remark 6. Are all base relations satisfiable in each model? Of course not, since each linear order is a tree in which all non-linear base relations are not satisfiable. The pictorial representations in table 3 and table 4, however, suggest that there is a tree consisting of seven nodes in which all base relations are satisfiable.

Table 4. Refining the 19 base relations

Symbol	Relation	Defining Constraints	Pictorial Representation
pb (*pbi*)	*I* partially before *J*	$s_J \parallel e_I, \exists t(s_I \prec t \wedge t \prec e_I \wedge t \prec s_J)$	
a (*ai*)	*I* adjacent to *J*	$s_I \prec s_J, s_J \parallel e_I, \neg\exists t(s_I \prec t \wedge t \prec e_I \wedge t \prec s_J)$	
po (*poi*)	*I* partially overlaps *J*	$s_I \prec s_J, e_I \parallel e_J, \exists t(s_J \prec t \wedge t \prec e_I \wedge t \prec e_J)$	
pm (*pmi*)	*I* partially meets *J*	$s_I \prec s_J, s_J \prec e_I, e_I \parallel e_J, \neg\exists t(s_J \prec t \wedge t \prec e_I \wedge t \prec e_J)$	
ps	*I* partially starts *J*	$s_I = s_J, e_I \parallel e_J, \exists t(s_I \prec t \wedge t \prec e_I \wedge t \prec e_J)$	
t	*I* touches *J*	$s_I = s_J, e_I \parallel e_J, \neg\exists t(s_I \prec t \wedge t \prec e_I \wedge t \prec e_J)$	

Ten base relations are definable by interval endpoints in branching time if we allow for quantifying over nodes. The table refines some of the entries in Table 3, namely *ib*, *im*, and *ie*.

Remark 7. How does satisfiability of base relations depend on the splitting type of the tree at hand? We will not give a complete and satisfactory answer to this question, but some comments will help to illustrate the problem. Consider first a $\mathbb{Q}$-tree that branches only at Dedekind gaps of $\mathbb{Q}$. Then, for example, the relation *pm* is satisfiable in that tree only if interval terms denote left-open intervals. For $\mathbb{R}$-trees of splitting type 1 the problem is less crucial since one can show that each interpretation of interval variables in such a tree can be transformed into an equivalent closed-interval interpretation. The same does not hold true for $\mathbb{R}$-trees of type 2. For example, let $\mathcal{B}$ be any tree of this kind that has at least two branches. Consider then the sentence "*I* meets both of the two disjoint intervals *J* and *J′*", which can be expressed by $I\, m\, J \wedge I\, m\, J' \wedge \neg J\, ps\, J'$. Although this formula is satisfiable in $\mathcal{B}$, it is not satisfiable by a closed-interval interpretation in that tree.

4 The Neighborhood Graph

Assume that two intervals are related by one of the base relations presented in the previous section. What happens if we move one of the intervals (in very small "steps") along one of the branches in which it is contained? Which relation could hold between the intervals if we increased or decreased one of them? These questions are usually answered by presenting a neighborhood graph. The neighborhood graph is also often understood as a similarity measure for the conceptual neighborhood of relations. This technique was first discussed by Freksa [12].

In what follows we will only investigate the first of these two questions, i. e., we enforce the size persistency constraint. Obviously, the neighborhood graph for interval relations in branching time (cf. Fig. 6) contains the corresponding graph for linear time, which is depicted in Fig. 5.

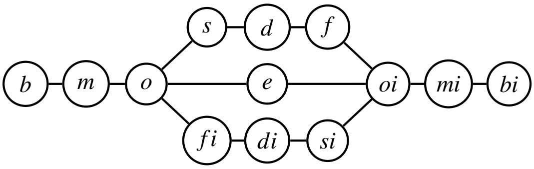


Fig. 5. The neighborhood graph of the interval algebra for linear time

In contrast to the neighborhood graph for linear time, the graph for branching time is not unique, i. e., it makes a significant difference for the neighborhood graph whether we fix one interval (for instance the second relatum) and allow for moving the other interval, or if we admit that at each step one of both intervals can be moved. A neighborhood graph with the first property will be referred to as a neighborhood graph of *type 1*. Neighborhood graphs in the weaker sense are said to be of *type 2*. It is immediately clear that the neighborhood graph of type 1 is a base of the neighborhood graph of type 2. Moreover, neighborhood graphs of type 2 are symmetric in the following sense: If the graph contains a transition between two relations, then it also contains a transition between the converses of these relations. Thus, the neighborhood graph of type 2 can be obtained from the type 1 graph by closing the latter one under converses.

It is remarkable that the neighborhood graph of type 1 is more precise, since it encodes the relation transitions that are admitted if only the interval at the first relation argument is allowed to be moved. To our knowledge, there is no other neighborhood graph in the literature that shows a similar behavior. The different types of neighborhood graphs mirror the underlying tree structure. Furthermore, in the neighborhood graph of type 2 each linear relation except f, fi, and e has at least one non-linear neighbor. Hence, if I and J are related by one of these relations and if the tree at hand is branching

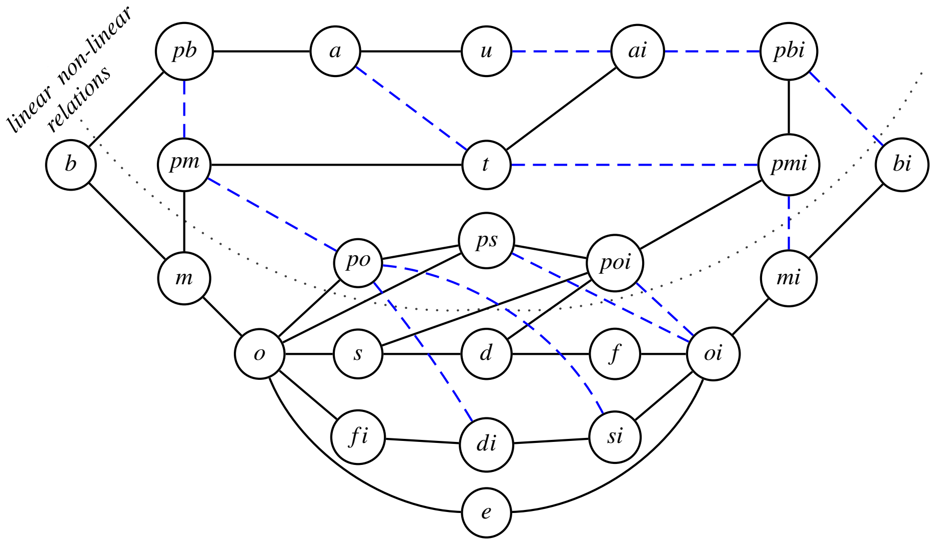


Fig. 6. The neighborhood graph of branching time interval relations. The undashed lines represent the relation transitions that are possible if the first relatum is moved, while the second is fixed. Dashed lines represent those relation transitions that are admitted additionally if one of both relata can be moved

dense, then we can move J in such a way that, afterwards, I and J are related by a non-linear relation. In the cases f, fi, and e, we always have to move J first in a position that is linearly related to I.

A number of non-linear relations have only non-linear relations as neighbors, namely u, t, a, and ai. If two intervals are related by one of these relations, then they have to be contained in distinct branches that split before one of the intervals ends. Thus, in each of these situations an interval cannot be moved in one step such that it is completely contained in the branch of the other interval.

5 Computational Complexity

Assume that a set of relations between some intervals is given. One question might be whether this set of relations is consistent. In other words: Is it possible to construct a tree in which all these relations are satisfiable? What is the computational effort to construct such a tree? Another interesting question might be whether the new, non-linear relations of branching time, especially the “partial relations”, provide additional complexity and whether the general satisfiability problem is still NP-complete (cf. [33]).

Complexity analyses for the point algebra of branching time have been carried out by Broxvall [6]. Broxvall shows that the complexity of the satisfiability problem for the point algebra, which is in P for linear time models, is NP-complete for branching time models. Moreover, Broxvall also identified five maximal tractable subclasses. One of these is the class $\Gamma_A = \{\prec, \preceq, \prec\succ, \prec{=}\succ, \|, {=}\|, =, \neq, \prec\|, \preceq\|\}$, which will be of interest in what is to follow.[4]

We start by introducing some semantical concepts in a more formal way. The constraint language of the point algebra PA_{br} consists of infinitely many (point) variables $v \in V(PA_{\mathrm{br}})$ and a relation symbol for each of the 16 relations of PA_{br} (we will not sharply distinguish relations and relation symbols, etc., provided the meaning is clear from the context). Formulae of this language (called *PA_{br} constraints*) are expressions of the form $v\,r\,v'$, where v and v' are variables and $r \in PA_{\mathrm{br}}$. A *PA_{br} model* is an ordered pair $\mathcal{M} = \langle \mathcal{B}, a \rangle$ consisting of a tree $\mathcal{B} = \langle T, \prec \rangle$ and a (variable) assignment $a\colon V(PA_{\mathrm{br}}) \longrightarrow T$. The satisfiability relation is defined in the natural way, namely $\mathcal{M} \models v \prec v'$ if and only if $a(v) \prec a(v')$, etc. A constraint set of PA_{br} (i. e., a finite set of PA_{br} constraints), C, is said to be *satisfiable* if there exists a PA_{br} model such that $\mathcal{M} \models \phi$ for each $\phi \in C$.

Analogously, the constraint languages of the interval algebras IA^{19}_{br} (cf. Table 3) and IA^{24}_{br} (cf. Table 4) are introduced. An *IA_{br} model* is an ordered pair $\mathcal{M} = \langle \mathcal{B}, a \rangle$, where $\mathcal{B} = \langle T, \prec \rangle$ is a tree and a is a map that assigns to each interval variable I a closed interval $a(I) = [s_I, e_I]$ of $\mathcal{B}$. The model relation for IA^{19}_{br} and IA^{24}_{br} is introduced as sketched in Table 3 and in Table 4, respectively. For example,

$$\mathcal{M} \models I\ pb\ J \iff s_J \parallel e_I \text{ and there is a } t \in T \text{ with } s_I \prec t, t \prec e_I, \text{and } t \prec s_J.$$

It is now an easy exercise to prove the following proposition:

Proposition 8. *Let C be a constraint set of IA^{19}_{br} or IA^{24}_{br}. Equivalent are:*

(i) *C is satisfiable.*

(ii) *C is satisfiable in a finite tree.*

(iii) *C is satisfiable in a $\mathbb{Q}$-tree (of type 1 or type 2).*

(iv) *C is satisfiable in an $\mathbb{R}$-tree (of type 1 or type 2).* □

It is important to note that these equivalences hold only if we restrict consideration to closed-interval interpretations.

In the remainder of the section we will focus on the algebra IA^{19}_{br}. We aim at showing that reasoning with IA^{19}_{br} relations is NP-complete. To show this we define first a mapping, Φ, that translates constraint sets of IA^{19}_{br} containing base relations only into constraint sets of PA_{br} (it is obvious how this translation could be extended to arbitrary IA^{19}_{br} relations). First we partition the set of PA_{br} variables, $V(PA_{\mathrm{br}})$, into two infinite subsets S and E. Let $V(IA_{\mathrm{br}}) \to S,\ I \mapsto s_I$, and $V(IA_{\mathrm{br}}) \to E,\ I \mapsto e_I$, be injective mappings. According to the defining constraints in Table 3 we translate interval relation constraints as follows:

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[4] Here and in what follows we will use symbols such as "$=\|$" to denote the union of the relations $=$ and $\|$ (cf. section 2).

$$\begin{aligned}
\Phi(I\,b\,J) &:= \{s_I \prec e_I, e_I \prec s_J, s_J \prec e_J\}\\
\Phi(I\,m\,J) &:= \{s_I \prec e_I, e_I = s_J, s_J \prec e_J\}\\
&\vdots\\
\Phi(I\,e\,J) &:= \{s_I \prec e_I, s_J \prec e_J, s_I = s_J, e_I = e_J\}\\
\Phi(I\,ib\,J) &:= \{s_I \prec e_I, s_I \prec s_J, s_J \prec e_J\}\\
\Phi(I\,im\,J) &:= \{s_I \prec s_J, s_J \prec e_I, s_J \prec e_J\}\\
\Phi(I\,ie\,J) &:= \{s_I = s_J, s_I \prec e_I, s_J \prec e_J\}\\
\Phi(I\,u\,J) &:= \{s_I \prec e_I, s_J \prec e_J, s_I \parallel s_J\}
\end{aligned}$$

Finally we set

$$\Phi(\{\phi_1,\ldots,\phi_n\}) := \bigcup_{i=1}^{n} \Phi(\phi_i).$$

It is now easy to prove the following lemmata:

Lemma 9. *Let $\mathcal{M} = \langle \mathcal{B}, a\rangle$ be a finite IA_{br} model, C be a constraint set of IA_{br}^{19}, and $V(C)$ be the set of interval variables occurring in C. If $\mathcal{M}$ satisfies C, then there exists an assignment $\breve{a}\colon V(PA_{br}) \longrightarrow \mathcal{B}$ such that $\langle \mathcal{B}, \breve{a}\rangle$ satisfies $\Phi(C)$.* □

Lemma 10. *Let $\mathcal{M} = \langle \mathcal{B}, a\rangle$ be a finite PA_{br} model and let C be a constraint set of IA_{br}^{19} such that $\mathcal{M}$ satisfies $\Phi(C)$. Let $\hat{a}\colon V(IA_{br}) \longrightarrow \mathcal{B}$ be defined by $\hat{a}(I) := [a(s_I), a(e_I)]$. Then $\langle \mathcal{B}, \hat{a}\rangle$ satisfies C.* □

Thus, we obtain as a result:

Proposition 11. *Let C be a constraint set of IA_{br}^{19}. Then C is satisfiable if and only if $\Phi(C)$ is satisfiable.* □

We are now ready to present a complexity result for constraint satisfaction problems in IA_{br}^{19}. The general satisfiability problem in IA_{br}^{19} is defined as follows:

Given: A constraint set C of IA_{br}^{19}.
Question: Does there exist any (finite) tree $\mathcal{B}$ in which all constraints of C are satisfiable?

To discuss this problem in more detail, let us first consider the analogous problem for PA_{br}. Broxvall [6] showed that the general constraint satisfaction problem for PA_{br} is NP-hard. But he also presented a polynomial time algorithm *Branch* that is sound and complete for the subset Γ_A of point relations in branching time. Broxvall's algorithm takes as input a problem instance of the form $\Pi = (V, C)$, where V is a set of variables and C is a set of point constraints between these variables. The algorithm then continues as follows: First it partitions the set of variables into distinct components such that only variables related by a linear relation fall into the same component. Hence, on each such component one obtains a (total) linear order in a natural manner. In the next step the algorithm computes the subgraphs, followed by a computation of the transitive and reflexive closure of all these subgraphs. Finally, the root nodes of all components are identified, and the subgraphs are merged to a tree via these root nodes. As the partitioning was defined such that unrelated points are contained in distinct components, for each pair of branches, there exists exactly one node where the branches are merged. In the case of acceptance, the algorithm *Branch* outputs a tree satisfying the input constraint set C.

Lemma 12 (Broxvall [6]). *The algorithm Branch runs in polynomial time and is sound and complete for* Γ_A. □

To discuss the constraint satisfactions problem for IA^{19}_{br}, we consider first *basic constraint sets*. A constraint set C is said to be *basic* if C contains only base relations and if for each pair of interval variables I and J occurring in C, C contains exactly one constraint $I\,r\,J$. A *scenario*, then, is a model of a basic constraint set.

Then the satisfiability of a given basic constraint set of IA^{19}_{br}, C, can be decided by the following algorithm *IntervalBranch*:

1. Translate the interval constraint set C into the point constraint set $\Phi(C)$ of PA_{br}.
2. Compute the transitive closure $\Phi(C)^*$ of $\Phi(C)$.
3. Apply Broxvall's algorithm *Branch* on $\Phi(C)^*$.

In fact, the algorithm *IntervalBranch* is polynomial, and it is sound and complete for basic constraint sets of IA^{19}_{br}. This is an immediate consequence of Proposition 11, Lemma 12, and the following observation:

Lemma 13. *For each basic constraint set C of* IA^{19}_{br}*, the constraint set* $\Phi(C)^*$ *is basic too. Moreover, in* $\Phi(C)^*$ *only relations of the set* $\{\prec, \preceq, =, \|\} \subseteq \Gamma_A$ *occur.* □

Corollary 14. *Testing satisfiability of basic* IA^{19}_{br} *constraint sets is tractable.* □

Theorem 15. *The general satisfiability problem of* IA^{19}_{br} *is* NP*-complete.*

Proof. NP-hardness follows straightforward by a reduction from 3-colorability: Let $G = (V, E)$, $V = \{v_1, ..., v_n\}$ be an instance of 3-colorability. Then we use the following interval symbols $\{v_1, ..., v_n, c_1, c_2, c_3\}$ with the following constraints:

$$\begin{array}{ccll} c_1 & \{m\} & c_2 & \\ c_2 & \{m\} & c_3 & \\ v_i & \{m, e, mi\} & c_2 & \\ v_i & \{b, m, mi, bi\} & v_j & \forall (v_i, v_j) \in E \end{array}$$

It is immediately clear that this reduction is polynomial. Now it is obvious that the constraint system is satisfiable if and only if G can be colored with 3 colors. Therefore the problem is NP-hard.

Membership in NP follows from the following short description of a non-deterministic algorithm: Let C be a constraint set of IA^{19}_{br}. Guess (non-deterministically) a scenario for C. Since each scenario is basic, we only need to check, whether this scenario satisfies C. But this can be done in polynomial time, which is shown in Corollary 14. □

6 The Composition Table

A composition table for the interval algebra of branching time was first presented by Anger, Ladkin, and Rodriguez [4]. In this section we briefly discuss the differences between the composition table for branching time and that for linear time. This means that we focus on the composition of linear relations in the context of branching time.[5]

Table 5. Composing linear relations with linear relations in branching time

	b	*m*	*o*	*d*	*s*
bi	*I*	*ai*, *poi*, *bi*, *mi*, *oi*, *f*, *d*, *pbi*, *pmi*	*poi*, *bi*, *mi*, *oi*, *f*, *d*, *pbi*, *pmi*	*poi*, *bi*, *mi*, *oi*, *f*, *d*, *pbi*, *pmi*	*poi*, *bi*, *mi*, *oi*, *f*, *d*, *pbi*, *pmi*
mi	*a*, *po*, *b*, *m*, *o*, *fi*, *di*, *pb*, *pm*	*ps*, *s*, *si*, *e*	*pmi*, *poi*, *oi*, *d*, *f*	*pmi*, *poi*, *oi*, *d*, *f*	*pmi*, *poi*, *oi*, *d*, *f*
oi	*pb*, *pm*, *po*, *b*, *m*, *o*, *fi*, *di*	*pm*, *po*, *o*, *di*, *fi*	*ps*, *s*, *si*, *e*, *po*, *o*, *di*, *fi*, *poi*, *oi*, *d*, *f*	*poi*, *oi*, *d*, *f*	*poi*, *oi*, *d*, *f*
di	*pb*, *pm*, *po*, *b*, *m*, *o*, *fi*, *di*	*pm*, *po*, *o*, *di*, *fi*	*po*, *o*, *di*, *fi*	*ps*, *s*, *si*, *e*, *po*, *o*, *di*, *fi*, *poi*, *oi*, *d*, *f*	*po*, *o*, *di*, *fi*
si	*pb*, *pm*, *po*, *b*, *m*, *o*, *fi*, *di*	*pm*, *po*, *o*, *di*, *fi*	*po*, *o*, *di*, *fi*	*poi*, *oi*, *d*, *f*	*t*, *ps*, *s*, *si*, *e*

I is defined as the union of all basic *linear* relations.

First note that the composition table presented in Table 5 has to be read in the consistency-based sense (cf. [5]). This means that, in the general case (without further requirement on the class of intended models), the composition table cannot be read extensionally. Composition is understood extensionally if the algebraic function of

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[5] The complete composition table is available at `ftp://ftp.informatik.uni-freiburg.de/documents/papers/ki/ragni-woelfl-brallen-comp.pdf`. It is worth noting that Reich [28] corrected one of the entries in the original table.

composition of relations (as used in relation algebras) coincides with its set-theoretical characterization. More precisely, if $\mathcal{K}$ is the class of intended models, then composition can be read extensionally w. r. t. $\mathcal{K}$ if and only for each model $\mathcal{M}$ in $\mathcal{K}$ and for each "composition" r_{ij} of relations r_i and r_j (as indicated in the composition table), it holds:

$$\mathcal{M} \models x r_{ij} y \leftrightarrow \exists z (x r_i z \wedge z r_j y).$$

The consistency-based reading only requires the right-to-left implication in this equivalence. Recall that the composition tables for points or intervals in linear time can be read extensionally for the class of dense linear orders, but not for the class of finite orders.

Composition of interval relations for branching time cannot be read extensionally even for the class of dense flows of time. For example, though $ps \circ u \subseteq u$ is true in that class, $ps \circ u \supseteq u$ is not. The latter subsumption entails that for each pair of intervals with $I\,u\,K$, there exists an interval J such that $I\,ps\,J$ and $J\,u\,K$. But such an interval could only be found if a splitting point can be found in interval I that witnesses the existential quantification in the definition of the relation *ps*. Note that Anger, Ladkin, and Rodriguez [4] restrict consideration on branching dense trees. Presumably, this (very strong) requirement is owed to the extensional reading of composition. But to our knowledge, it has not yet been proved that, for the class of dense and branching dense trees, composition can always be read extensionally.

Is it possible that the composition of linear relations admits non-linear relations? For answering this question consider the following situation. Let I, J, and K be intervals such that $I\,r\,J$, $J\,r'\,K$, and both relations r and r' are linear Allen relations. It is immediately clear that the start points of I and J, respectively, are comparable with respect to $\prec$ and that the same holds true for the start points of J and K. But the start points of I and K need not be comparable. As an example of this situation, assume that interval J is before I and K and that intervals I and K start after a splitting point of two branches b and b' containing I and J, respectively.

In this situation a non-linear relation can occur only if the endpoint of interval J, e_J, is before the endpoint of I and the endpoint of K. The node e_J has to be before the endpoints e_I and e_K because e_I and e_K are unrelated. But the only linear relations where e_J is before e_I are the relations *bi*, *mi*, *oi*, *di*, and *si*. An analogous statement holds true for intervals J and K. For this reason composition of two linear base relations r and r' can admit a non-linear relation only if $r \in \{bi, mi, oi, di, si\}$ and $r' \in \{b, m, o, d, s\}$. The relations f, fi, and e imply the equality of the endpoints of two intervals and therefore they enforce that all three intervals are related by one of the linear relations.

7 Summary and Outlook

Starting from the question of how qualitative reasoning in non-linear time differs from reasoning in linear time, we first investigated non-linear relations between intervals in branching time. We showed that a naïve Allen representation of intervals is problematic in branching time contexts. To put this in other words, the boundary structure of intervals in branching time is more sophisticated than it is for intervals in linear time. Moreover, we saw that both the boundary structure of intervals and the splitting type of the tree at hand may influence the set of base relations that can be distinguished reasonably. But since our remarks on this topic are still somewhat sketchy, a more detailed analysis needs to be done.

In a second step we indicated that different algebras for intervals in branching time can be defined, depending on whether one allows for quantifying over tree nodes or not. From a cognitive point of view, it could be interesting to see if people actually distinguish between these two conceptual schemata.

This leads to another interesting question: The neighborhood graph for interval relations in branching time shows how interval relations can change in time. But does this graph also mirror a *conceptual* neighborhood of the presented relations? For example, it seems that each linear relation is conceptually more similar to a given linear relation than is each non-linear relation. Moreover, we saw that the underlying tree structure implied two different kinds of neighborhood graphs, which is remarkable since most neighborhood graphs known in the literature are unique. The neighborhood graph presented here did enforce size persistency. It will be interesting to compare this graph with a neighborhood graph figuring transitions of relations between increasing or decreasing intervals.

We investigated the complexity of the general satisfiability problem for the interval algebra IA_{br}^{19} and showed that it is NP-complete. Of course, an analogous result for IA_{br}^{24} would be desirable, but seems harder.

The complexity considerations are strongly related with the question of how to reason with intervals in branching time. Therefore, we analyzed the composition table of branching time interval relations and presented a “heuristics” explaining why the composition of two linear base relations may contain a non-linear base relation. We aim at showing the correctness of this table by a computer-aided proof.

Acknowledgments

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