A Categorical Perspective on Qualitative Constraint Calculi

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Abstract. In the domain of qualitative constraint reasoning, a subfield of AI which has evolved in the last 25 years, a large number of calculi for efficient reasoning about space and time has been developed. Reasoning problems in such calculi are usually formulated as constraint satisfaction problems. For temporal and spatial reasoning, these problems often have infinite domains, which need to be abstracted to (finite) algebras in order to become computationally feasible.
Ligozat [13] has argued that the notion of *weak representation* plays a crucial rôle: it not only captures the correspondence between abstract relations (in a relation algebra or non-associative algebra) and relations in a concrete domain, but also corresponds to algebraically closed constraint networks.
In this work, we examine properties of the category of weak representations and treat the relations between partition schemes, non-associative algebras and concrete domains in a systematic way. This leads to the notion of *semi-strong* representation, which captures the correspondence between abstract and concrete relations better than the notion of weak representation does. The slogan is that semi-strong representations avoid unnecessary loss of information. Furthermore, we hope that the categorical perspective will help in the future to provide new insights on the important problem of determining whether algebraic closedness decides consistency of constraint networks.

1 Introduction

Qualitative reasoning aims at describing the common-sense background knowledge on which our human perspective on the physical reality is based. Methodically, qualitative constraint calculi restrict the vocabulary of rich mathematical theories dealing with temporal or spatial entities such that specific aspects of these theories can be treated within decidable fragments with simple qualitative (i. e., non-metric) languages. Contrary to mathematical or physical theories about space and time, qualitative constraint calculi allow for rather inexpensive reasoning about entities located in space and time. For this reason, the limited expressiveness of qualitative representation formalism calculi is a benefit if such reasoning tasks need to be integrated in applications. For example, some of these calculi may be implemented for handling spatial GIS queries efficiently and some may be used for navigating, and communicating with, a mobile robot.

In the past 25 years the number of calculi for efficient reasoning about space and time has grown quite steadily. Examples of temporal calculi include the so-called point algebra, Allen's interval algebra [2], and Vilain's point-interval calculus [20]. The most prominent spatial calculi are mereotopological calculi (e. g., [3]), Frank's cardinal direction calculus [9], Freksa's double cross calculus [10], Egenhofer and Franzosa's 4- and 9-intersection calculi [7, 8], Ligozat's flip-flop calculus [14], and various region connection calculi proposed by Randell et al. [18], Cohn et al. [5], Düntsch et al. [6], and Gerevini and Renz [11].

Reasoning problems in qualitative calculi are usually formulated as so-called constraint satisfaction problems. Starting from a set of base relations (i. e., a family of relations that partitions the set of all tuples of domain elements), a constraint is a formula of the form xRy with variables x and y (taking values in given domains D_x and D_y) and a set of base relations R defined between the domains of x and y — the set of base relations, R, is read disjunctively and hence expresses imprecise knowledge about the concrete scenario described by the constraint formula. The constraint satisfaction problem with respect to a fixed qualitative calculus is to determine for a given constraint network (i. e., a finite set of constraints) whether there exists an assignment to its variables such that all constraints of the network become true. Further typical reasoning tasks are to check that some constraint is entailed by a constraint network, and to compute an equivalent minimal constraint network (all these reasoning tasks are equivalent under polynomial Turing reductions).

A crucial aspect for developing efficient algorithms for qualitative spatial and temporal reasoning is the fact that the underlying model classes usually contain infinite models. Hence, in order to test satisfiability of constraint networks, it is not feasible to enumerate all models and all possible assignments to variables in a fixed model until one finds a satisfying assignment. For this reason other techniques (such as path-consistency algorithms) must be applied for testing satisfiability. These techniques usually depend on the so-called composition table of the calculus at hand (for an example, see section 3). The idea behind these methods is to encode domain-dependent knowledge in a table that lists which relations may possibly hold between two objects a and b, when knowledge about the relations of a resp. b to some other objects is available.

However, there are different possibilities of how to read these composition tables [4]. And as a result of a somehow conceptual confusion, the path consistency method has sometimes been employed, although the underlying interpretation of composition was not justified by the given domain. To clarify this confusion, Ligozat [12] introduced the notion of *weak representation*, which not only captures the correspondence between abstract relations (in a relation algebra or non-associative algebra) and relations in a concrete domain, but also corresponds to algebraically closed constraint networks.

In this paper, we provide an even more abstract, namely a category-theoretical, point of view in order to examine properties of the weak representations and treat the relations between partition schemes, non-associative algebras, and concrete domains in a systematic way (some initial category-theoretic treatment is given by Ligozat [13]). This leads to the notion of *semi-strong* representation, which, in our opinion, captures the correspondence between abstract and concrete relations better than the notion of weak representation does.

The paper is organized as follows. In section 2, we will provide the basic category-theoretical background. In section 3, we discuss the category of partition schemes and show how qualitative calculi may be represented in a category-theoretical framework. Then in section 4, the category of non-associative relation algebras is introduced. Section 5 and section 6 are dedicated to discuss the relationships between strong and weak composition as well as those between weak and strong representations of abstract (non-associative) relation algebras. In this section we will also discuss a new reasonable notion of representation, which is slightly weaker than the concept of strong representation, but which is conceptually more adequate than the concept of weak representation.

2 Categorical Background

To start with, let us briefly recall some basic notions of category theory which we will refer to in the following sections (for useful introductory texts see [16, 1]).

A *category* $\mathbf{C}$ consists of a class $\mathrm{Ob}(\mathbf{C})$ of *objects*, for each pair (A,B) of objects a class $\mathrm{hom}_{\mathbf{C}}(A,B)$ of *morphisms*, where we write $f : A \to B$ for $f \in \mathrm{hom}(A,B)$, a choice of an *identity* $id_A : A \to A$ for each object A, and a *composition* operation assigning to each pair (g, f) of morphisms $f : A \to B$ and $g : B \to C$ its *composite* $g \circ f : A \to C$. These data are subject to two equational laws: composition is *associative*, i. e., $h \circ (g \circ f) = (h \circ g) \circ f$ whenever these terms are defined, and identities are neutral w. r .t. composition, i. e., $id_B \circ f = f = f \circ id_A$ for each morphism $f : A \to B$.

Given two categories $\mathbf{A}$ and $\mathbf{B}$, a *functor* $F : \mathbf{A} \to \mathbf{B}$ assigns to each $\mathbf{A}$-object A a $\mathbf{B}$-object FA, and to each $\mathbf{A}$-morphism $f : A \to B$ a $\mathbf{B}$-morphism $Ff : FA \to FB$ such that identities and composition are preserved, i. e., $Fid_A = id_{FA}$ for each object A, and $F(g \circ f) = Fg \circ Ff$ whenever $g \circ f$ is defined.

Example 1. The category **Set** of sets and maps has as its objects all sets, and as morphisms $X \to Y$ all maps $f : X \to Y$. Composition is the usual composition of maps, and identities are identity maps. **Set** has several *non-full* subcategories (a subcategory $\mathbf{A}$ of a category $\mathbf{B}$ is called *full* if $\mathrm{hom}_{\mathbf{A}}(A,B) = \mathrm{hom}_{\mathbf{B}}(A,B)$ whenever $A,B \in \mathrm{Ob}\,\mathbf{A}$), such as the category of sets and injective maps and the category $\mathbf{Set}_{surj}$ of sets and surjective maps.

The category **Top** of topological spaces has as its objects all topological spaces, and as morphisms $X \to Y$ all continuous maps $f : X \to Y$. **Top** is a typical example of a *concrete* category, given in terms of structured objects over a base category, in this case **Set**. Formally, one has a *forgetful functor* $\mathbf{Top} \to \mathbf{Set}$ that maps every topological space to its underlying set and each continuous map to the map itself.

An important strength of the definition of category is that it is stable under numerous constructions yielding new categories, such as the following. Given a category $\mathbf{C}$, its *dual* $\mathbf{C}^{op}$ is the category having the same objects as $\mathbf{C}$, and as morphisms $A \to B$ the $\mathbf{C}$-morphisms $B \to A$. Given functors $F : \mathbf{A} \to \mathbf{C}$ and $G : \mathbf{B} \to \mathbf{C}$, the *comma category* (F,G) has triples (A, f, B) as objects, where $A \in \mathrm{Ob}\,\mathbf{A}$, $B \in \mathrm{Ob}\,\mathbf{C}$, and $f : FA \to GB$, and pairs (g,h) as morphisms $(A_1, f_1, B_1) \to (A_2, f_2, B_2)$, where $g : A_1 \to A_2$, $h : B_1 \to B_2$, and $Gh \circ f_1 = f_2 \circ Fg$. An important special case of the latter construction is that F is a *constant* functor taking all objects to a fixed object $C \in \mathrm{Ob}\,\mathbf{C}$, and all morphisms to id_C. In this case, the comma category (F,G) is denoted more simply as (C,G), and its description simplifies as follows: objects are pairs (f,B), where $B \in \mathrm{Ob}\,\mathbf{B}$ and $f : C \to GB$, and morphisms $(f_1,B_1) \to (f_2,B_2)$ are morphisms $h : B_1 \to B_2$ such that $Gh \circ f_1 = f_2$.

Given two functors $F,G : \mathbf{A} \to \mathbf{B}$, a *natural transformation* $\eta : F \to G$ consists of a family $(\eta_A : FA \to GA)_{A \in \mathbf{A}}$ of $\mathbf{B}$-morphisms, such that the following naturality condition is fulfilled: for each $\mathbf{A}$-morphism $f : A_1 \to A_2$, $Gf \circ \eta_{A_1} = \eta_{A_2} \circ Ff$.

3 Partition Schemes

As stated before, qualitative calculi usually start from a set of so-called *base relations*, which is a family of relations that partitions the set of all tuples of domain elements at hand. This approach can be formally captured by the notion of *partition scheme*.

Definition 1 (cp. Ligozat and Renz [15]). Let U be a non-empty set. A *partition scheme* on U is defined by a finite (index) set I with a distinguished element $i_0 \in I$, a unary operation $\smile$ on I, and a family of binary relations $(R_i)_{i \in I}$ on U such that

(a) $(R_i)_{i \in I}$ is a partition of $U \times U$ in the sense that the R_i are pairwise disjoint and jointly exhaustive.[3]
(b) R_{i_0} is the diagonal relation $\{(x,x) \mid x \in U\}$.
(c) $R_{i^\smile}$ is the (set-theoretical) converse of relation R_i, for each $i \in I$.

The relations R_i are referred to as *basic relations*. In the following we often write

$$U \times U = \bigcup_{i \in I} R_i$$

to denote partition schemes.

Here, we additionally introduce *morphisms of partition schemes*.

Definition 2. Let $(R_i)_{i \in I}$ and $(S_j)_{j \in J}$ be partition schemes on U and V, respectively. A *morphism* $(h,k)\colon U \times U = \bigcup_{i \in I} R_i \longrightarrow V \times V = \bigcup_{j \in J} S_j$ is a pair of functions $h\colon U \longrightarrow V$ and $k\colon I \longrightarrow J$ such that

(a) $k(i_0) = j_0$,
(b) $k(i^\smile) = k(i)^\smile$, and
(c) $(h \times h)[R_i] \subseteq S_{k(i)}$, i. e., for all $x,y \in U$ with $x\,R_i\,y$, $h(x)\,S_{k(i)}\,h(y)$ holds.

If such a morphism exists, we also say that $(V \times V = \bigcup_{i \in J} S_j)$ is *refined* (via (h,k)) to $(U \times U = \bigcup_{i \in I} R_i)$ (note that the source scheme is the target of the refinement).

[3] Ligozat and Renz require the R_i to be non-empty. This requirement leads to the problem that, for example, the RCC8 calculus leads to a partition scheme only for certain (e.g. connected) topological spaces. We hence drop this requirement in this paper.

Together with pairwise identities and composition, this gives a category **Part** of partition schemes.

Example 2. RCC5 is a functor $RCC5\colon \mathbf{Set}_{surj}^{op} \longrightarrow \mathbf{Part}$, where $\mathbf{Set}_{surj}$ is the category of sets and surjective maps. A set S is sent to the partitioning of $\mathcal{P}(S) \times \mathcal{P}(S)$ into the five relations DR, PO, PP, PPi, and EQ (cf. Table 1). A surjective function $f\colon S_1 \longrightarrow S_2$ is mapped to (f^{-1}, id). For such an f, f^{-1} preserves emptiness, and by surjectivity of f also non-emptiness, of sets. Since set-theoretic intersection and difference commute with f^{-1}, f^{-1} preserves the RCC5 relations (cf. Table 1).

Table 1: Characteristic properties of RCC5 relations. n means "non-empty", e means "empty", ? means "don't care".

$relation$	$X_1 \cap X_2$	$X_1 \setminus X_2$	$X_2 \setminus X_1$
PO	n	n	n
DR	e	n	n
EQ	?	e	e
PP	?	e	n
PPi	?	n	e

Example 3. RCC8 is a functor $RCC8\colon \mathbf{Top}_{surj,open}^{op} \longrightarrow \mathbf{Part}$, where $\mathbf{Top}_{surj,open}$ is the category of topological spaces and surjective open continuous maps. Let $(S, \mathcal{O})$ be such a topological space and let *Reg* denote the set of all non-empty regular closed subsets. The functor sends $(S, \mathcal{O})$ to the partitioning of $Reg \times Reg$ into the relations PO, EQ, DC, EC, TPP, NTPP, TPPi, and NTPPi (cf. Table 2). A surjective continuous and open function $f\colon (S_1, \mathcal{O}_1) \longrightarrow (S_2, \mathcal{O}_2)$ is mapped to (f^{-1}, id). Then f^{-1} commutes with *int* (additionally to the properties listed in Example 2), and by Table 2, f^{-1} preserves the RCC8 relations.

Table 2: Characteristic properties of RCC8 relations. n means "non-empty", e means "empty", ? means "don't care".

$relation$	$X_1 \cap X_2$	$X_1 \setminus X_2$	$X_2 \setminus X_1$	$int(X_1) \cap int(X_2)$	$X_1 \setminus int(X_2)$	$X_2 \setminus int(X_1)$
PO	n	n	n	n	n	n
EC	n	n	n	e	n	n
DC	e	n	n	e	n	n
EQ	n	e	e	n	?	?
TPP	n	e	n	n	n	n
$NTPP$	n	e	n	n	e	n
$TPPi$	n	n	e	n	n	n
$NTPPi$	n	n	e	n	n	e

Example 4. The refinement of $RCC5$ into $RCC8$ is a natural transformation $\theta\colon RCC8 \to RCC5 \circ \mathcal{U}$, where $\mathcal{U}\colon \mathbf{Top}_{surj} \longrightarrow \mathbf{Set}_{surj}$ is the forgetful functor. θ just forgets regular closedness of sets, and sends each RCC8 relation to the corresponding RCC5 relation (e.g. both TPP and NTPP are sent to PP). The homomorphism property of $\theta_{(S,O)}$ follows from the fact that Table 1 is part of Table 2.

4 Non-Associative Algebras

We are interested in approximating compositions of relations in a finite way. Ligozat and Renz [15] consider so-called non-associative relation algebras in order to capture weak composition (as introduced in the following section).

Definition 3 (Maddux [17]). A *non-associative (relation) algebra* is a tuple $\mathcal{A} = (A, +, -, \cdot, 0, 1, ;, ^{\smile}, \Delta)$ such that:

1. $(A, +, -, \cdot, 0, 1)$ is a Boolean algebra.
2. Δ is a constant, $^{\smile}$ a unary and ; a binary operation such that, for any $a, b, c \in A$:

$$(a)\ (a^{\smile})^{\smile} = a \qquad (b)\ \Delta;a = a;\Delta = a \qquad (c)\ a;(b+c) = a;b + a;c$$
$$(d)\ (a+b)^{\smile} = a^{\smile} + b^{\smile} \quad (e)\ (a-b)^{\smile} = a^{\smile} - b^{\smile} \quad (f)\ (a;b)^{\smile} = b^{\smile};a^{\smile}$$
$$(g)\ (a;b)\cdot c^{\smile} = 0 \text{ if and only if } (b;c)\cdot a^{\smile} = 0$$

Given non-associative algebras A and B, a *homomorphism* from A to B is a homomorphism $h\colon A \longrightarrow B$ on the underlying Boolean algebras such that

(a) $h(\Delta) \geq \Delta$,
(b) $h(a^{\smile}) = h(a)^{\smile}$ for all $a \in A$, and
(c) $h(a;b) \geq h(a);h(b)$ for all $a, b \in A$.

Together with set-theoretic identities and composition, this defines the category **NA** of non-associative algebras.

A non-associative algebra is a *relation algebra* if the operation ; is associative. Let **RA** denote the full subcategory of **NA** consisting of all relation algebras. A homomorphism of non-associative algebras is *strong* if the above inequalities (a) and (c) are equalities. Let $\mathbf{NA}_s$ be the (non-full) subcategory of **NA** consisting of the strong homomorphisms.

An *atomic* non-associative algebra is one that is atomic as a Boolean algebra. The atoms are also called *basic relations* in this case.

5 Strong and Weak Composition

In our setting, *strong* composition can be modelled by a contravariant functor $\mathbf{S}\colon \mathbf{Set}^{op} \to \mathbf{RA}$. On objects, it maps a set U to $\mathcal{P}(U \times U)$ equipped with the usual set-theoretic interpretation of $+$ as union, $-$ as set difference, $\cdot$ as intersection, 0 as empty relation, 1 as the universal relation $U \times U$, ; as composition, $^{\smile}$ as converse, and Δ as the diagonal relation. Given a function $f\colon U \longrightarrow V$, $\mathbf{S}(f)$ takes inverse images w.r.t. $f \times f$, i. e., a relation $R \subseteq V \times V$ is taken to $\mathbf{S}(f)(R) = (f \times f)^{-1}[R]$. By abuse of notation, we will denote the composition of $\mathbf{S}$ with the inclusion of $\mathbf{RA}$ into $\mathbf{NA}$ by $\mathbf{S}\colon \mathbf{Set}^{op} \longrightarrow \mathbf{NA}$. The drawback of this construction is that the (usually infinite) space of all relations is not structured. This structuring can be obtained by using a partition scheme.

The notion of weak composition approximates composition of set-theoretic relations (which is strong composition) within a partition scheme, thus leading to a non-associative algebra. The functor $\mathbf{W}\colon \mathbf{Part}^{op} \longrightarrow \mathbf{NA}$ maps a partition scheme

$$U \times U = \bigcup_{i \in I} R_i$$

to the non-associative algebra that has $\mathcal{P}(I)$ as its Boolean algebra component. The converse is given by pointwise application of $\smile$; the diagonal is i_0. Composition is given by *weak composition*:[4]

$$I_1 ; I_2 = \{ i \mid i_1 \in I_1, i_2 \in I_2, (R_{i_1} \circ R_{i_2}) \cap R_i \neq \emptyset \}.$$

Given a morphism (h,k) of partition schemes, $\mathbf{W}(h,k)$ is just k^{-1}. In order to prove that this is a homomorphism of non-associative algebras, note that $(R_{i_1} \circ R_{i_2}) \cap R_i \neq \emptyset$ implies $(S_{k(i_1)} \circ S_{k(i_2)}) \cap S_{k(i)} \neq \emptyset$.

6 Weak, Strong and Semi-Strong Representations

We now discuss representations of abstract non-associative algebras in concrete (i.e. set-theoretic) domains. With the above machinery, we are able to recast the definition of Ligozat and Renz [15] as follows:

Definition 4. A *weak representation* of a non-associative algebra A (in a domain with underlying set U) is a homomorphism of non-associative algebras $\varphi\colon A \longrightarrow \mathbf{S}(U)$. A weak representation is *diagonal-persevering*, if $\varphi(\Delta) = \Delta$.

Note that our notion of weak representation is slightly weaker than that in [13], because we do only require that $\varphi(\Delta)$ contains the diagonal relation, while [13] requires weak representations to be always diagonal-preserving.

Proposition 1. *Given two weak representations $\varphi, \psi\colon A \longrightarrow \mathbf{S}(U)$ with*

$$\varphi(a) \subseteq \psi(a) \ (a \in A),$$

we already have

$$\varphi = \psi.$$

Proof. Writing $\overline{a}$ for the complement $1 - a$ of a, we have for $a \in A$ that $\varphi(a) = \varphi(\overline{\overline{a}}) = \overline{\varphi(\overline{a})}$, and similarly $\psi(a) = \overline{\psi(\overline{a})}$. Since $\varphi(\overline{a}) \subseteq \psi(\overline{a})$, we get $\overline{\psi(\overline{a})} \subseteq \overline{\varphi(\overline{a})}$. But this means $\psi(a) \subseteq \varphi(a)$. Altogether, $\varphi(a) = \psi(a)$. □

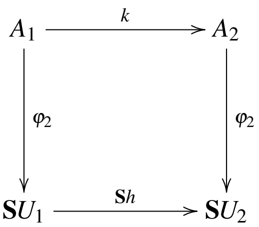
[4] Note that it is common in the category theory community to use $\circ$ for function composition in applicative order, and ; for diagrammatic order. By contrast, in the qualitative reasoning community both $\circ$ and ; are used for composition of *relations* in *diagrammatic* order. $\circ$ stands for the usual set-theoretic composition, ; for weak composition.

Definition 5 (cp. Ligozat [13]). Given weak representations $\varphi\colon A \longrightarrow \mathbf{S}(U)$ and $\psi\colon A \to \mathbf{S}(V)$, a *morphism* $h\colon \varphi \longrightarrow \psi$ *of weak representations* is a function $h\colon U \longrightarrow V$ such that for all $a \in A$[5] and $(x,y) \in U \times U$, if $(x,y) \in \varphi(a)$, then $(h(x),h(y)) \in \psi(a)$.

Proposition 2. *A morphism* $h\colon \varphi \longrightarrow \psi$ *of weak representations equivalently is a function* $h\colon U \longrightarrow V$ *such that* $\varphi = \mathbf{S}(h) \circ \psi$*. That is, the category of weak representations* $\mathbf{WR}(A)$ *is the comma category* $(A, \mathbf{S})$.

Proof. The implication $(x,y) \in \varphi(a) \Rightarrow (h(x),h(y)) \in \psi(a)$ is equivalent to $\varphi(a) \subseteq \mathbf{S}(h)(\psi(a))$. By Prop. 1, this is equivalent to $\varphi(a) = \mathbf{S}(h)(\psi(a))$. □

Definition 6. The category **WR** of weak representations (over *varying* non-associative algebras) is a comma category. Objects are weak representations $\varphi\colon A \longrightarrow \mathbf{S}(U)$, and morphisms are commutative squares:



Theorem 1. *The functor* $\mathbf{W}\colon \mathbf{Part}^{op} \longrightarrow \mathbf{NA}$ *(introduced in section 4) can be extended to a functor* $\mathbf{W}\colon \mathbf{Part}^{op} \longrightarrow \mathbf{WR}$*, by regarding the non-associative algebra of a partition scheme as weakly represented in the underlying set of the partition scheme itself.*

Proof. $U \times U = \bigcup_{i\in I} R_i$ can be represented in $\mathbf{S}U$ by just mapping a set $J \subseteq I$ to $\bigcup_{j\in J} R_j$. This clearly is a homomorphism of Boolean algebras, preserves the diagonal as well as converse relations (by the definition of partition scheme), and weakly preserves composition by the definition of weak composition. Given a morphism $(h,k)\colon (U \times U = \bigcup_{i\in I} R_i) \longrightarrow (V \times V = \bigcup_{i\in J} S_j)$, let $\mathbf{W}(h,k)$ be $(k^{-1}, \mathbf{S}h)$. To prove that this is a morphism in **WR**, given $J_0 \subseteq J$, in light of Prop. 1 it suffices to show that

$$\bigcup_{i\in k^{-1}(J_0)} R_i \subseteq (h \times h)^{-1}\Big(\bigcup_{j\in J_0} S_j\Big).$$

But this easily follows from $(h \times h)[R_i] \subseteq S_{k(i)}$. □

A weak representation is *strong* if it is strong as a homomorphism of non-associative algebras. Unfortunately, weak representations arising from partition schemes are usually not strong. However, a weak representation postulates only a very loose connection between abstract and set-theoretic composition. Consider the following examples:

Example 5. Let *RCC*5 be the non-associative algebra of *RCC*5, U be a non-empty set, and $\varphi\colon RCC5 \longrightarrow \mathbf{S}U$ map the base relations as follows: *EQ* is mapped to $U \times U$, and the other base relations are mapped to $\emptyset$. This is easily extended to sets of base relations.

[5] Ligozat requires this only for basic relations, i.e. for the atoms of an atomic non-associative algebra. This seems to be unnaturally weak; in case of finite algebras, it is equivalent to our formulation.

Example 6. Let A be an atomic non-associative algebra such that composition and converse distributes over arbitrary joins (note that this holds in particular for any *finite* non-associative algebra), with Δ atomic (note that this implies $\Delta = \Delta^{\smile}$). Then we can define a non-associative algebra $Loss(A)$. $Loss(A)$ is like A, but with composition ; replaced by composition #, which is defined on basic relations as follows:

$$a\#b = \begin{cases} b, & \text{if } a = \Delta \\ a, & \text{if } b = \Delta \\ 1, & \text{if } a \neq \Delta \neq b, \Delta \leq a;b \\ 1-\Delta, & \text{otherwise} \end{cases}$$

By the distributivity assumption, the laws for non-associative algebras need only be verified for basic relations, which is not difficult.

Since $Loss(A)$ enlarges the composition of A, the identity is a homomorphism $id : Loss(A) \to A$. Hence, any weak representation of A leads to a weak representation of $Loss(A)$ by composing with $id : Loss(A) \to A$. In particular, we get weak representations of $Loss(RCC5)$ and $Loss(RCC8)$.

These weak representations are hardly useful for anything, because abstract composition only provides very little information about concrete composition. While Example 5 is in a sense pathological because the representation is not diagonal-preserving (and this is exploited in an extreme way), the representations of Example 6 *are* diagonal-preserving. In this example, abstract compositions are larger than necessary. Indeed, most information about concrete composition is thrown away, and only information about the diagonal relation is kept. In the light of the possibility to have a better weak representation (namely the standard representations for RCC5, RCC8 etc.), this must be considered as an unneccessary loss of information. Therefore, we cannot agree with the slogan of [15] “A qualitative calculus is a weak representation.” Apparently, a qualitative calculus has a connection between its abstract non-associative algebra and its concrete domain that is stronger than the one described in terms of weak representations. Hence, we will strengthen the representation condition as follows, in order to capture the situation that no unnecessary loss of information occurs:

Definition 7. Given an atomic non-associative algebra A, a weak representation $\varphi : A \to \mathbf{S}(U)$ is said to be *semi-strong* if for all $b, c \in A$,

$$b;c = \bigvee \{a \mid a \text{ atomic}, (\varphi(b) \circ \varphi(c)) \cap \varphi(a) \neq 0\}.$$

While the weak representations induced by RCC5 and RCC8 (Examples 2 and 3) are semi-strong, the weak representations of Examples 5 and 6 are not. The notion of semi-strong representation thus avoids the inclusion of representations that have only a limited connection between the abstract algebra and the concrete representation, while simultaneously providing more flexibility than strong representations, which are too strong to capture weak composition. Indeed, semi-strong representations are precisely the notion that captures weak composition:

Observation 1. *The weak representation induced by a partition scheme is semi-strong.*

Proof. Obvious, by definition of weak composition. □

Recall from [19] that a (finite) constraint network on an atomic non-associative algebra A is a pair $\mathcal{N} = (N, \rho)$, where N is a (finite) set of nodes (or variables) and ρ a map $\rho : N \times N \to A$. For each pair (i, j) of nodes, $\rho(i, j)$ is the constraint on the arc (i, j). A network is atomic if ρ is in fact a map into the set of atoms of A. It is normalized if $\forall i, j \in N,\ \rho(i, j) = \Delta$ if $i = j$, and $\forall i, j \in N,\ \rho(j,i)^{\smile} = \rho(i, j)$. A network (N', ρ') is a refinement of (N, ρ) if $\forall i, j \in N$ we have $\rho'(i, j) \leq \rho(i, j)$. Finally, a network is algebraically closed, or a-closed, if $\forall i, j, k \in N,\ \rho(i, j) \leq \rho(i,k);\rho(k, j)$. (Note that a network can be made a-closed using the path-consistency algorithm.)

The crucial observation of Ligozat and Renz [15] is the following: Given a normalized and atomic constraint network $\mathcal{N}$ over a non-associative relation algebra A, $\mathcal{N}$ is a-closed if and only if it corresponds to a weak representation $\rho^{\mathcal{N}}$ in $\mathbf{WR}(A)$. Now an a-closed network $\mathcal{N}$ is consistent w.r.t. a given domain of interpretation $\varphi \in \mathbf{WR}(A)$ if and only if there is a morphism of weak representations $h\colon \varphi \longrightarrow \rho^{\mathcal{N}}$. This can be summarized as follows:

Observation 2. *For a weak representation that is weakly terminal*[6] *in* $\mathbf{WR}(A)$, *a-closedness decides consistency of constraint networks.*

Ligozat and Renz [15] point out that the question whether a-closedness decides consistency of constraint networks is of fundamental nature, and they give a criterion to determine whether a qualitative calculus enjoys this property. Still, it may be hard to apply their criterion in practice. With our categorical approach via weakly terminal objects, we can try to apply standard categorical, algebraic and coalgebraic methods for answering the question whether a-closedness decides consistency.

7 Conclusion

We have outlined a categorical framework for the unifying treatment of (binary) qualitative constraint calculi. In particular, we have introduced a category of partition schemes and defined standard calculi such as RCC5 and RCC8 as functorial indexings of partition schemes. Moreover, we have identified the category of weak representations of non-associative algebras as a comma category over the category of non-associative algebras, and we have proposed a strengthening of the notion of weak representation: *semi-strong* representations capture the properties of weak composition more precisely than both weak and strong representations do. They allow coarser abstractions than strong representations, but avoid unnecessary loss of information (that can occur with weak representations). We suggest to strengthen the slogan of Ligozat and Renz [15] “A qualitative calculus is a weak representation” in the following way:

A qualitative calculus is a semi-strong representation.

This paper could only present the main ideas of a category-theoretical approach to qualitative constraint calculi. The following questions still remain open:

[6] A *weakly terminal object* of a category $\mathbf{C}$ is an object T in $\mathbf{C}$ such that to each object A in $\mathbf{C}$, there exists a morphism $A \to T$.

- Weakly terminal representations in $\mathbf{WR}(A)$ have the pleasant property that a-closedness decides consistency of constraint networks. Does this characterize weak terminality?
- Given a non-associative algebra, what are its weakly terminal representations? This question, however, is of more theoretical interest, because usually, one does not start with a non-associative algebra, but rather with a concrete domain.
- Hence, the following question is more important: Given a partition scheme, if we apply the functor $\mathbf{W}\colon \mathbf{Part}^{op} \longrightarrow \mathbf{WR}$ to it, how to determine whether the resulting semi-strong representation is weakly terminal? And if it is not weakly terminal, can it be embedded somehow in a semi-strong and weakly terminal representation (possibly using a refinement of the partition scheme)? We hope to apply algebraic and coalgebraic methods to tackle this problem.

A further perspective is indicated by the fact that we had to restrict the functorial representation of typical calculi such as RCC5 and RCC8 to not entirely natural non-full subcategories, e.g. the category of surjective open continuous maps in the case of RCC8. The main suspect as the cause of this technical difficulty is the disjointness requirement in the definition of partition scheme, which forces the use of base relations such as "proper part of" and which itself is motivated by constructions of representations where the base relations play the role of atoms. A technically more pleasing alternative might be to choose a more natural set of base relations, compatible with preimage formation in the relevant category (this would classify relations such as "part of" or "interior part of' as natural, but 'proper part of" as unnatural), from which other relations may be built as Boolean combinations. One may then hope to obtain representations using newly constructed atoms, in analogy to the use of maximally consistent sets in propositional logics.

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