The Finest of its Class: The Natural, Point-Based Ternary Calculus $\mathcal{LR}$ for Qualitative Spatial Reasoning

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Abstract. We develop a theory for ternary point-based calculi such that the relations are invariant when all points are mapped by rotations, scalings or translations and propose methods to determine arbitrary transformations and compositions of such relations. We argue that calculi based on such relation systems should satisfy two criteria. First, the relation system should be closed under transformations, compositions and intersections and have a finite base that is jointly exhaustive and pairwise. This implies that the well-known path consistency algorithm can be used to conclude implicit knowledge without any loss of information. If this is the case, we call the calculus *practical*. Second, we say that a relation system is *natural* if all relations and their complements give rise to sets of points that are connected. The main result of the paper is then the identification of a maximally refined calculus amongst the practical natural RST calculi, which turns out to be very similar to Ligozat's flip-flop calculus. From that it follows, e.g., that there is no finite refinement of the TPCC calculus by Moratz et al that is closed under transformations, composition, and intersection.

1 Introduction

Reasoning about spatial configurations is an important task for many applications such as geographical information systems (GIS), natural language understanding, and for automatic geometric proofs. In many cases, no detailed quantitative information of the spatial structures under consideration is available. For example, images show only the relative alignment of objects, and text information often contains rough descriptions such as "coming from point a you have to turn right at point b to reach region c". In such cases, qualitative approaches that define formal representations of everyday descriptions are used. They are an effective way to conclude implications of the given spatial information.

In recent years, a series of qualitative spatial calculi have been proposed and analyzed, such as a calculus for reasoning about topological relations [1, 13], calculi about orderings [5, 9, 14], directions [2, 12], relative position of a point with respect to a line segment [8, 3, 4, 11] and others.

In this paper, we will focus on calculi as the latter ones, whereby we will develop a general theory for ternary, point-based relations that are invariant when all points are mapped by rotations, scalings or translations. This means that, e.g., the claim that one point lies on a straight line between two other points remains true when the whole map is rotated, shifted or the scale is changed. We will call such relations ***RST relations*** according to the first letters of the transformations. We present a new way of describing them that gives each RST relation a standard name or representation, and we calculate compositions and transformations of qualitative relations.

Starting with the observation that Freksa's double-cross calculus [3, 4] has some unsatisfactory properties [16], we consider the entire class of RST calculi, of which the double-cross calculus is an instance. For practical reasons, we expect from a calculus that it is finite and closed under certain fundamental operations such as transformations and compositions that are used to conclude implicit knowledge. It is known that Freksa's calculus and its finite refinements do not have this closure property [16]. Furthermore, we require that the relation systems of the calculus has the property that base relations do not denote arbitrary sets of points, but they should be connected regions. The goal of this paper is to identify the most refined RST calculus with the these properties. As it turns out, this is a calculus which is very close to Ligozat's flip-flop calculus, and which we call $\mathcal{LR}$.

The remainder of the paper is structured as follows. In the next section we define the calculus $\mathcal{LR}$ as an example for a calculus with ternary RST relations. In Section 3, we define the requirements necessary for a practical calculus and prove some basic properties of all ternary calculi not yet dealing with the RST property. In Section 4, we introduce the theory of RST Calculi, the way of describing ternary RST relations and some fundamental consequences. In Section 5, we prove that $\mathcal{LR}$ is the finest finite RST calculus with the properties proposed in Section 3. In Section 6, we conclude.

2 The Calculus $\mathcal{LR}$

When human beings or robots proceed along a path (going from point a_1 to point a_2), they will always distinguish between things ahead of them or behind their back, and they can distinguish whether objects they pass have been to their right or to their left, or if they have directly met them. Moreover, it is easy to recognize in which order certain points have been reached, hence, which objects are between others on the line. However, without additional information, it is often not possible to find out in which distance the objects are from the path, or at which angle precisely something can be seen.

All these spatial expressions involve the current standpoint or ***starting point*** a_1, the walking direction and a focus point a_3. The direction can be easily represented by a goal or ***reference point*** a_2 along the line. Now we can describe the spatial situation by sentences such as (refer to Figure 1.a):

1. Looking from a_1 to a_2, a_3 is to the left.
2. Walking from a_1 to a_2, $a_3{}'$ is always at your back.

This idea will now be formalized by introducing relations and operations on the relations. We will call it the left-right-distinguishing calculus $\mathcal{LR}$.

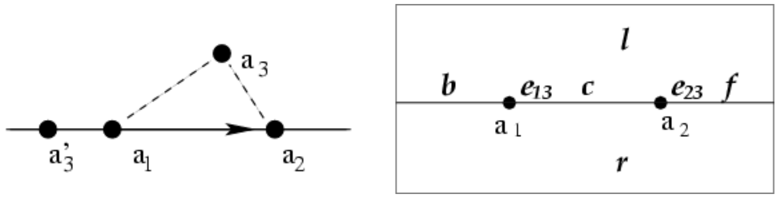

Fig. 1. a. The two examples of relations in the text

b. The base relations of $\mathcal{LR}$ where $a_1 \neq a_2$. given The letters are explained in Table 1.

2.1 Relations

For each of the different situations we introduce a relation with three arguments a_1, a_2, a_3. Each argument represents a point in the plane $\mathbb{R}^2$. We consider two alternatives in the case that a_1 and a_2 coincide. We distinguish between seven situations for a_3 when a_1 and a_2 are different. This leads to a calculus that contains the nine relations as shown in Table 1. Note that for all possible triples of points there is a relation of the calculus (see Figure 1.b), which distinguishes this calculus from Ligozat's flip-flop calculus which does not have the relations $\mathfrak{eq}$ and $\mathfrak{e}_{12}$.

$\mathcal{LR}$ Base Relation for triple (a_1, a_2, a_3)		angle at a_1 $\angle(a_2, a_1, a_3)$	angle at a_2 $\angle(a_3, a_2, a_1)$	Meaning
$\mathfrak{eq}$	$a_1 = a_2 = a_3$	-	-	a_1, a_2 and a_3 are all equal.
$\mathfrak{e}_{12}$	$a_1 = a_2 \neq a_3$	-	-	a_3 is different from $a_1 = a_2$.
$\mathfrak{e}_{13}$	$a_1 = a_3 \neq a_2$	-	0^o	a_2 is different from $a_1 = a_3$.
$\mathfrak{e}_{23}$	$a_1 \neq a_2 = a_3$	0^o	-	a_1 is different from $a_2 = a_3$.
	$a_1 \neq a_2 \neq a_3 \neq a_1$:			Looking from a_1 to a_2:
$\mathfrak{b}$	back	180^o	0^o	a_3 is back behind a_1.
$\mathfrak{c}$	closer	0^o	0^o	a_3 is closer to a_1 than a_2.
$\mathfrak{f}$	further	0^o	180^o	a_3 is further ahead.
$\mathfrak{r}$	right	$]-180^o; 0^o[$	$]-180^o; 0^o[$	a_3 is to the right.
$\mathfrak{l}$	left	$]0^o; 180^o[$	$]0^o; 180^o[$	a_3 is to the left.

Table 1. Definition of the base relations of $\mathcal{LR}$

To express uncertainty, all unions of these nine relations are included in the calculus $\mathcal{LR}$. For example, we write $(a_1, a_2, a_3) \in \mathfrak{eq} \cup \mathfrak{e}_{12}$ if we know that $a_1 = a_2$ but do not know anything about a_3. The union of all relations is denoted by $\top$. $(a_1, a_2, a_3) \in \top$ contains no information about (a_1, a_2, a_3).

Definition 1 (Base relations)
We call a minimal subset of relations $\mathcal{B} \subset \mathcal{C}$ a ***base*** *of the calculus $\mathcal{C}$ if any relation* R

in $\mathcal{C}$ is a set union of some relations in $\mathcal{B}$. The relations in $\mathcal{B}$ are called ***base relations*** *of $\mathcal{C}$.*

2.2 Operations

A calculus provides formal ways to conclude implicit knowledge. There are several ways to derive new claims. Therefore, we define operations on the relations so that the resulting relations represent what we can derive. As standard methods for ternary relations we use intersection, transformation and composition which are generalizations of the corresponding well-known operations for binary relations[7, 17] and of the operations on ternary relations as defined by Isli et al [5].

Intersection: As with binary relations, if we have two claims about the same triple, we can combine them into one claim.

$$(a_1, a_2, a_3) \in \mathsf{R}_1 \quad \wedge \quad (a_1, a_2, a_3) \in \mathsf{R}_2 \Rightarrow (a_1, a_2, a_3) \in (\mathsf{R}_1 \cap \mathsf{R}_2)$$

Transformation: Transformation generalizes the binary concept of the converse (exchanging the arguments of a binary relation). If we have a claim about a triple, we can derive a claim about any permutation of the triple. Therefore, we define the transformation operations as follows:

Definition 2 (Transformation of a relation)

Let $\pi \in \mathfrak{S}_3$ be a permutation of the positions of a ternary relation $\mathsf{R} \subset (\mathbb{R}^2)^3 := \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, then R^π is the **transformed relation** *with the property*

$$(a_1, a_2, a_3) \in \mathsf{R} :\Longleftrightarrow \bar{\pi}((a_1, a_2, a_3)) := (a_{\pi(1)}, \ldots, a_{\pi(3)}) \in \mathsf{R}^\pi$$

The operation T^π: $\mathsf{R} \mapsto \mathsf{R}^\pi$ is called a transformation.

Examples:

The permutation (231) is a rotation of the indexes that maps 2 to 3, 3 to 1 and 1 to 2. $T^{(231)}$ corresponds with Isli's [5] "rotation" operation.

$$(a_1, a_2, a_3) \in \mathsf{f} \iff (a_2, a_3, a_1) \in \mathsf{b} = \mathsf{f}^{(231)}$$

The permutation (23) is an exchange of the last two indexes that maps 1 to 1, 2 to 3 and 3 to 2. $T^{(23)}$ corresponds with Isli's "converse" operation.

$$(a_1, a_2, a_3) \in \mathsf{l} \cup \mathsf{e}_{12} \iff (a_1, a_3, a_2) \in \mathsf{r} \cup \mathsf{e}_{13} = (\mathsf{l} \cup \mathsf{e}_{12})^{(23)}$$

Table 2 displays all the transformations of $\mathcal{LR}$ (see also [6]).

Composition: Restrictions that originate from a combination of the relations of two overlapping triples are called composition. With ternary relations, one can think of several ways of composing them, depending on the number and order of overlapping points. We proved [15] that the only case in which proper new restrictions can be derived is when the two relations concerning different triples have two common points. Depending on the order of the points in the original relations, we distinguish six different types of composition of which two examples are shown in Figure 2.

$\mathcal{LR}$ R	$\mathsf{R}^{(12)}$	$\mathsf{R}^{(13)}$	$\mathsf{R}^{(23)}$	$\mathsf{R}^{(231)}$	$\mathsf{R}^{(321)}$
eq	eq	eq	eq	eq	eq
e_{12}	e_{12}	e_{23}	e_{13}	e_{23}	e_{13}
e_{13}	e_{23}	e_{13}	e_{12}	e_{12}	e_{23}
e_{23}	e_{13}	e_{12}	e_{23}	e_{31}	e_{12}
b	f	c	b	c	f
c	c	b	f	f	b
f	b	f	c	b	c
r	l	l	l	r	r
l	r	r	r	l	l

Table 2. Transformations of $\mathcal{LR}$

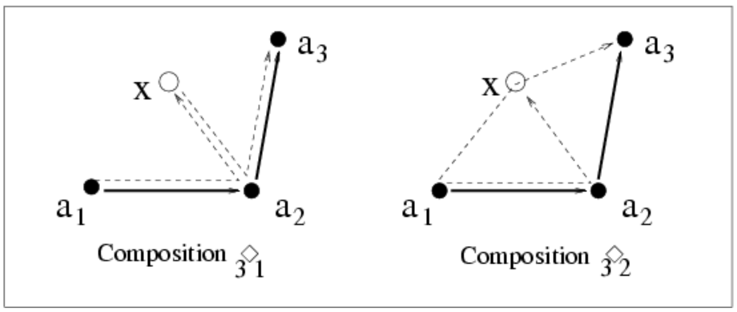

Fig. 2. The idea of compositions. The dotted lines indicate the relations R, S and the solid line the relation $\mathsf{R} \diamond \mathsf{S}$

Definition 3 (Compositions) *Let* $\mathsf{R}, \mathsf{S} \subset (\mathbb{R}^2)^3$ *be ternary relations. Then for* $\kappa \neq \lambda$ *and* $\kappa, \lambda \in \{1, 2, 3\}$ *the composition* $_\kappa\diamond_\lambda$ *is defined as follows:*

$$(a_1, a_2, a_3) \in \mathsf{R}\ {}_3\diamond_2\ \mathsf{S} :\Longleftrightarrow \exists x : (a_1, a_2, x) \in \mathsf{R} \wedge (a_1, x, a_3) \in \mathsf{S}$$

$$(a_1, a_2, a_3) \in \mathsf{R}\ {}_3\diamond_1\ \mathsf{S} :\Longleftrightarrow \exists x : (a_1, a_2, x) \in \mathsf{R} \wedge (x, a_2, a_3) \in \mathsf{S}$$

$$(a_1, a_2, a_3) \in \mathsf{R}\ {}_2\diamond_3\ \mathsf{S} :\Longleftrightarrow \exists x : (a_1, x, a_3) \in \mathsf{R} \wedge (a_1, a_2, x) \in \mathsf{S}$$

$$(a_1, a_2, a_3) \in \mathsf{R}\ {}_2\diamond_1\ \mathsf{S} :\Longleftrightarrow \exists x : (a_1, x, a_3) \in \mathsf{R} \wedge (x, a_2, a_3) \in \mathsf{S}$$

$$(a_1, a_2, a_3) \in \mathsf{R}\ {}_1\diamond_3\ \mathsf{S} :\Longleftrightarrow \exists x : (x, a_2, a_3) \in \mathsf{R} \wedge (a_1, a_2, x) \in \mathsf{S}$$

$$(a_1, a_2, a_3) \in \mathsf{R}\ {}_1\diamond_2\ \mathsf{S} :\Longleftrightarrow \exists x : (x, a_2, a_3) \in \mathsf{R} \wedge (a_1, x, a_3) \in \mathsf{S}.$$

Example:

Suppose there is a house x to the left of the path from a_1 to a_2, and there is a tree a_3 behind the house as seen from a_1. Then the tree a_3 is also to the left of the path since

$$(a_1, a_2, x) \in l,\ (a_1, x, a_3) \in f \Rightarrow (a_1, a_2, a_3) \in l\ {}_3\diamond_2\ f = l.$$

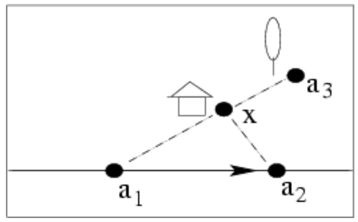

Fig. 3. An example for applying transformation and composition

When looking from a_2 to the house, the tree a_3 will be seen to the right. To derive this, transformation and composition are needed:

$$(a_1, a_2, x) \in \mathsf{l} \Rightarrow (a_2, x, a_1) \in \mathsf{l}^{(231)} = \mathsf{l}.$$

Note that $\mathsf{l}\ {}_3\diamond_1\ \mathsf{f} = \mathsf{r}$, hence

$$(a_2, x, a_1) \in \mathsf{l}, (a_1, x, a_3) \in \mathsf{f} \Rightarrow (a_2, x, a_3) \in \mathsf{r}.$$

Throughout this paper we will revisit this calculus along with the new concepts we introduce.

3 Properties for Ternary Calculi

3.1 Closure Properties

In $\mathcal{LR}$, any given spatial constellation gives rise to a corresponding set of relations between its triples, and it is possible to describe all conclusions that can be derived using intersection, transformation and composition of the relations. It is reasonable to generalize these properties as requirements for calculi since this is what calculi are used for.

A calculus $\mathcal{C}$ is called closed under an operation iff for any choice of relations of $\mathcal{C}$ the result of the operation is again a relation of $\mathcal{C}$.

We assume that all calculi are closed under set union to represent uncertainty. If a calculus is closed under intersection, all transformations and all compositions, then implicit information can be made explicit using the conclusion methods of the calculus mentioned above.

In general, like in the case of $\mathcal{LR}$, we expect that for any triple of points of the plane $\mathbb{R}^2$ there is a unique base relation that describes it. In other words, all base relations jointly cover the set of all possible triples $(\mathbb{R}^2)^3$, and the intersection of different base relations is disjoint. Hence we have a jointly exhaustive pairwise disjoint basis (JEPD basis). A calculus with a JEPD basis is closed under intersection and set complement. In order to be part of a representation in a computer, this set of base relations has to be finite. Any practical calculus should satisfy these requirements.

Definition 4 (Practical calculus)
*A calculus $\mathcal{C}$ will be called **practical** if it is closed under transformations, compositions and intersections and has a finite JEPD basis.*

Practical calculi have advantageous formal properties. A variation of the well-known path-consistency algorithm [10] can be used to find inconsistencies. In this section, we will present some useful algebraic results that we need for the purpose of this paper: The main result is that there is a tight dependence between composition and transformation: Each composition can be derived from any of the others using transformations. Hence it is sufficient to store one composition table. Moreover, transformations and compositions distribute over set union, hence in a practical calculus it is sufficient to define these operations on the base relations.

To state the dependence property, it is necessary to note that the concatenation of transformations corresponds with concatenation of their underlying permutations [15]. As a consequence we have the following result:

Remark 1 (Inverse Transformation) *For each transformation operation (T^{π}), there exists the inverse transformation operation $(T^{\pi})^{-1} \ := \ T^{\pi^{-1}}$:*

$$(T^{\pi})^{-1}(T(\mathsf{R})) \ = \ T^{\pi\circ\pi^{-1}}(\mathsf{R}) \ = \ T^{id}(\mathsf{R}) \ = \ \mathsf{R}.$$

The inverse transformation operation is used to derive one composition from another:

Proposition 1 (Interdependence of compositions)
For all $\kappa_1, \lambda_1, \kappa_2, \lambda_2 \in \{1, 2, 3\}$:

$$\mathsf{R} \ {}_{\kappa_2}\!\diamond_{\lambda_2} \ \mathsf{S} \ = \ (\mathsf{R}^{\pi} \ {}_{\kappa_1}\!\diamond_{\lambda_1} \ \mathsf{S}^{\pi})^{\pi^{-1}},$$

where $\pi \in \mathfrak{S}$ is the permutation for which: $\pi(\kappa_1) \ = \ \kappa_2, \ \ \pi(\lambda_1) \ = \ \lambda_2$

Proof:
We introduce the notation $s_{\kappa}(x)((a_1, a_2, a_3))$ for the substitution of the κ-th element in the triple by x.

First we prove an equality that holds for any transformation operation π and composition ${}_{\kappa}\!\diamond_{\lambda}$:

$$T^{\pi}(\mathsf{R} \ {}_{\kappa}\!\diamond_{\lambda} \ \mathsf{S}) \ = \ T^{\pi}(\mathsf{R}) \ {}_{\pi^{-1}(\kappa)}\!\diamond_{\pi^{-1}(\lambda)} \ T^{\pi}(\mathsf{S}) \qquad (*)$$

By the definition of transformation operation and composition we obtain

$$\begin{aligned}
& (a_1, a_2, a_3) \in T^{\pi}(\mathsf{R} \ {}_{\kappa_2}\!\diamond_{\lambda_2} \ \mathsf{S}) \\
\iff & (a_{\pi^{-1}(1)}, \ldots, a_{\pi^{-1}(3)}) \in (\mathsf{R} \ {}_{\kappa_2}\!\diamond_{\lambda_2} \ \mathsf{S}) \\
\iff & \exists\, d : (s_{\kappa_2}(d))(a_{\pi^{-1}(1)}, \ldots, a_{\pi^{-1}(3)}) \in \mathsf{R} \text{ and } (s_{\lambda_2}(d))(a_{\pi^{-1}(1)}, \ldots, a_{\pi^{-1}(3)}) \in \mathsf{S} \\
\iff & \exists\, d : (s_{\pi^{-1}(\kappa_2)}(d))((a_1, a_2, a_3)) \in \mathsf{R}^{\pi} \text{ and } (s_{\pi^{-1}(\lambda_2)}(d))((a_1, a_2, a_3)) \in \mathsf{S}^{\pi} \\
\iff & \exists\, d : (s_{\kappa_1}(d))((a_1, a_2, a_3)) \in \mathsf{R}^{\pi} \text{ and } (s_{\lambda_1}(d))((a_1, a_2, a_3)) \in \mathsf{S}^{\pi} \\
\iff & (a_1, a_2, a_3) \in (\mathsf{R}^{\pi} \ {}_{\kappa_1}\!\diamond_{\lambda_1} \ \mathsf{S}^{\pi})
\end{aligned}$$

which proves the equality (*).
With the premise on π that $\pi^{-1}(\kappa_1) = \lambda_1$ and $\pi^{-1}(\kappa_2) = \lambda_2$ follows:

$$\mathsf{R}\ {}_{\kappa_2}\!\diamond_{\lambda_2}\ \mathsf{S} = T^{\pi^{-1}}(T^{\pi}(\mathsf{R}\ {}_{\kappa_2}\!\diamond_{\lambda_2}\ \mathsf{S})) \overset{(*)}{=} T^{\pi^{-1}}(T^{\pi}(\mathsf{R})\ {}_{\kappa_1}\!\diamond_{\lambda_1}\ T^{\pi}(\mathsf{S})).$$

(qed.)

Proposition 2 (Distribution properties of practical calculi)
1. Let $\mathcal{C}$ be closed under transformation and set union. Then for any set of relations $\mathsf{R}_i \in \mathcal{C}$:

$$T^{\pi}(\bigcup_{i\in I} R_i) = \bigcup_{i\in I} T^{\pi}(R_i)$$

2. For each $0 < \kappa, \lambda \leq 3$ and relations R_i, S:

$$(\bigcup_{i\in I} \mathsf{R}_i)\ {}_{\kappa}\!\diamond_{\lambda}\ \mathsf{S} = \bigcup_{i\in I} (\mathsf{R}_i\ {}_{\kappa}\!\diamond_{\lambda}\ \mathsf{S})$$

and

$$\mathsf{S}\ {}_{\kappa}\!\diamond_{\lambda}\ (\bigcup_{i\in I} \mathsf{R}_i) = \bigcup_{i\in I} (\mathsf{S}\ {}_{\kappa}\!\diamond_{\lambda}\ \mathsf{R}_i)$$

Proof:

$$\begin{aligned}
\bar{\pi}\big((a_1,a_2,a_3)\big) \in T^{\pi}(\textstyle\bigcup_{i\in I} \mathsf{R}) &\iff (a_1,a_2,a_3) \in (\textstyle\bigcup_{i\in I} R_i) \\
&\iff \exists i \in I: \quad ((a_1,a_2,a_3)) \in R_i \\
&\iff \exists i \in I: \quad \bar{\pi}((a_1,a_2,a_3)) \in T^{\pi}(\mathsf{R}) \\
&\iff \bar{\pi}((a_1,a_2,a_3)) \in (\textstyle\bigcup_{i\in I} (T^{\pi}(\mathsf{R}))
\end{aligned}$$

For all κ, λ, R_i, S holds:

$$\begin{aligned}
& (a_1,a_2,a_3) \in (\textstyle\bigcup_{i\in I} \mathsf{R}_i)\ {}_{\kappa}\!\diamond_{\lambda}\ \mathsf{S} \\
\iff & \exists x: \ s_{\kappa}(x)(a_1,a_2,a_3) \in (\textstyle\bigcup_{i\in I} \mathsf{R}_i) \ \wedge\ s_{\lambda}(x)(a_1,a_2,a_3) \in \mathsf{S} \\
\iff & \exists x: \ (\textstyle\bigvee_{i\in I} (s_{\kappa}(x)(a_1,a_2,a_3) \in \mathsf{R}_i)) \ \wedge\ (s_{\lambda}(x)(a_1,a_2,a_3) \in \mathsf{R}_2) \\
\iff & \exists x: \ \textstyle\bigvee_{i\in I} ((s_{\kappa}(x)(a_1,a_2,a_3) \in \mathsf{R}_i) \ \wedge\ (s_{\lambda}(x)(a_1,a_2,a_3) \in \mathsf{R}_2)) \\
\iff & \textstyle\bigvee_{i\in I} (\exists x: ((s_{\kappa}(x)(a_1,a_2,a_3) \in \mathsf{R}_i) \ \wedge\ (s_{\lambda}(x)(a_1,a_2,a_3) \in \mathsf{R}_2))) \\
\iff & (a_1,a_2,a_3) \in \textstyle\bigcup_{i\in I} (\mathsf{R}_i\ {}_{\kappa}\!\diamond_{\lambda}\ \mathsf{S})
\end{aligned}$$

The second form of this property can be proved analogously. (qed.)

3.2 Refinement

Typically, in a real-world situation, the information available does not match one-to-one the relations of a calculus. We have included the operation set union so that we can describe spatial information for which we have only indefinite knowledge.

Sometimes, conversely, the knowledge available is more detailed than a selected calculus can express. Then, information is lost when building the knowledge base. This loss is smaller if there are more detailed relations available. Considering that some expressions are combinations of finer ones, e. g. "on the way from a to b" means either "at a", or "at b", or "in between", we come to the idea of refining some relations. A calculus that comprises another one's relations as unions of its own relations, will be called a refinement of the coarser one.

Definition 5 (Refinement)
*A calculus $\mathcal{C}_{fine}$ is called **finer** than another calculus $\mathcal{C}_{coarse}$ if each relation in $\mathcal{C}_{coarse}$ is a set union of relations of $\mathcal{C}_{fine}$. Then, $\mathcal{C}_{coarse}$ is called **coarser** than $\mathcal{C}_{fine}$.*

At the beginning of Section 5, we will introduce another concept, the concept of connectivity, and we will call a calculus natural if its base relations and their complements are connected. Our goal is to find the finest natural practical (hence: finite) calculus. We will concentrate on a class of calculi that proved to be of much interest because its relations are independent of scaling, rotation or shift from fixed reference points: RST calculi.

4 RST calculi

In this section, we will analyze the special properties of RST calculi and develop a theory of RST relations.

Definition 6 (RST calculus)
*Rotations, scalings with scaling factor > 0 and translations and concatenations of such mappings will be called **RST automorphisms**. A ternary relation R over points in the plane $\mathbb{R}^2$ is called **RST relation** if it has the **RST property**:*

For all RST automorphisms ρ holds

$$(a_1, a_2, a_3) \in \mathsf{R} \Rightarrow (\rho(a_1), \rho(a_2), \rho(a_3)) \in \mathsf{R}.$$

*If all relations of a calculus are RST relations, it is called an **RST calculus**.*

In the next subsections, we will see that there is a finest RST calculus. Thus arbitrary RST relations can be understood as union of the base relations of the finest RST calculus. Because of Proposition 2, the operations of transformation and composition of other RST calculi are based on the transformations and compositions of the finest one.

4.1 Standardized triples

Figure 1b. shows that for fixed points a_1, a_2 as starting and reference point, there are different relations of $\mathcal{LR}$, depending on the location of a_3 with respect to a_1 and a_2. All relations except eq and e_{12} are represented by a region in the plane that is a possible location for a_3 given the position of a_1, a_2. This scheme is the basis of the following idea: Each triple can be mapped by rotation, scaling and translation to a standardized triple (b_1, b_2, b_3) such that $b_1 = (0, 0)$ and $b_2 = (0, 0)$ (if a_1 and a_2 coincide), or $b_2 = (1, 0)$ (if $a_1 \neq a_2$).

We will now describe the function η that maps a triple to its standardized triple. This can most easily be done if we identify a point in $\mathbb{R}^2$ with a complex number using the standard isomorphism between $\mathbb{R}^2$ and $\mathbb{C}$, because the RST automorphisms of $\mathbb{R}^2$ are exactly those mappings that correspond with simple arithmetic operations in $\mathbb{C}$: The addition of a complex number corresponds with a translation, multiplication with a scalar value $r \in \mathbb{R}$ corresponds with a scaling, and multiplication with a purely imaginary number $r\imath$ ($r \in \mathbb{R}$) corresponds with a rotation of the complex plane.

To simplify the reading, for $z_i = x_i + \imath y_i \in \mathbb{C}$ let (z_1, z_2, z_3) denote the triple $((x_1, y_1), (x_2, y_2), (x_3, y_3))$. We distinguish three cases:

A. For $z_1 = z_2 = z_3$ (i.e. $(z_1, z_2, z_3) \in \mathcal{eq}$), we define the standardization as $\eta((z_1, z_2, z_3)) := (0, 0, 0)$. This corresponds with a shift of the plane.

B. For $z_1 = z_2 \neq z_3$ (i.e. $(z_1, z_2, z_3) \in \mathcal{e}_{12}$), set $\eta((z_1, z_2, z_3)) := (0, 0, 1)$.

C. For $z_1 \neq z_2$, set

$$\eta((z_1, z_2, z_3)) := (0, 1, \frac{z_3 - z_1}{z_2 - z_1}).$$

Note that these functions are well-defined.

In addition to a shorter notation, this $\mathbb{C}$ based representation is motivated by the following strategy of our proof: We show that any RST relation can be identified with a set of complex numbers. The advantage of such a notation is that we can determine complex functions that correspond with compositions and transformations on RST relations. By investigating consequences of possible results of these functions, we will find constraints on the set of relations in a practical calculus. By combining several of these constraints, we prove that the finest practical calculus cannot be further refined. Note that we will only use arithmetic properties of $\mathbb{C}$.

To motivate our first step, we show that triples have the same standardization η only if there is no RST relation that can distinguish between them.

Proposition 3 (Standardization)
$\eta((z_1, z_2, z_3)) = \eta((z_1', z_2', z_3'))$
iff there is an RST automorphism α *that maps* (z_1, z_2, z_3) *to* (z_1', z_2', z_3')
iff for all RST relations R *holds:* $(z_1, z_2, z_3) \in \mathsf{R} \Leftrightarrow (z_1', z_2', z_3') \in \mathsf{R}$.

Proof:
First we show: $\eta((z_1, z_2, z_3)) = \eta((z_1', z_2', z_3'))$ implies that there is an RST automorphism α with $\alpha(z_1, z_2, z_3)) = (z_1', z_2', z_3')$.

By definition, on (z_1, z_2, z_3), η is equal to a concatenation of the complex subtraction

$$\tau : z \mapsto z - z_1$$

(corresponding with a translation of points in $\mathbb{R}^2$) and the multiplication (depending on the case A., B. or C.)

$$\rho_A : z \mapsto 1 \cdot z \qquad \text{or} \qquad \rho_B : z \mapsto \frac{1}{z_3 - z_1} \cdot z \qquad \text{or} \qquad \rho_C : z \mapsto \frac{1}{z_2 - z_1} \cdot z$$

(corresponding with a rotation and scaling of $\mathbb{R}^2$). Similarly, there are τ', ρ' so that $\eta((z_1', z_2', z_3')) = \rho'(\tau'((z_1', z_2', z_3')))$. Hence $\alpha = (\tau')^{-1} \circ (\rho')^{-1} \circ \rho \circ \tau$ is the desired RST automorphism.

For the reverse direction (i.e. for each RST automorphism α: $\eta((z_1, z_2, z_3)) = \eta(\alpha((z_1, z_2, z_3)))$ holds), it is sufficient to show this property for each rotation-scaling

$\alpha : z \mapsto z \cdot c$ ($c \in \mathbb{C}$) and for each translation $\alpha : z \mapsto z + d$ ($d \in \mathbb{C}$). It is easy to verify that $\eta((z_1, z_2, z_3))\eta((z_1 \cdot c, z_2 \cdot c, z_3 \cdot c))$ and $\eta((z_1, z_2, z_3)) = \eta((z_1 + d, z_2 + d, z_3 + d))$.

Still we have to prove the second equivalence. For triples $a = (a_1, a_2, a_3),\ \ b = (b_1, b_2, b_3)$ regard the relation $a \sim_{RST} b$ that holds iff there is an RST automorphism α so that $\alpha(a) = b$. Because the inverse and concatenations of RST automorphisms are RST automorphisms, $\sim_{RST}$ is an equivalence relation, and the triples fall into corresponding equivalence classes.

By definition of $\sim_{RST}$, $\sim_{RST}$-classes and their set unions are RST relations. The RST property means that if an RST relation contains a triple of an equivalence class, it contains the whole class, hence each RST relation is a set union of equivalence classes. A triple belongs exactly to all RST relations that are supersets of its $\sim_{RST}$-equivalence class. This proves the claim. (qed.)

4.2 The finest RST calculus $\mathcal{F}$

The proof shows that RST relations are exactly the supersets of $\sim_{RST}$ equivalence classes. The RST calculus $\mathcal{F}$ whose base relations are the $\sim_{RST}$ equivalence classes hence is the finest RST calculus. Our goal is to describe the transformations and compositions of $\mathcal{F}$. Then we can derive any RST calculus from $\mathcal{F}$, as unions of $\mathcal{F}$'s transformations and compositions because of the Distribution Property 2. As we have seen, each equivalence class is characterized by its standardized triple. Almost all these triples only differ in its third number. To be able to calculate, we will identify each such triple $\eta(z_1, z_2, z_3) = (0, 1, {z_3}')$ with the single complex number ${z_3}' = \frac{z_3 - z_1}{z_2 - z_1}$. An RST relation R contains e_{12} or eq or consists of triples whose reference and starting point are different. In the latter case, RST relations can be represented and denoted by the region (set of points) in $\mathbb{C}$ $z_3 | \ (0, 1, {z_3}') \in \mathsf{R}$.

Definition 7 (Representability, RST relations as regions) *:*
Let $\mathsf{R} \subset (\top \setminus \mathsf{e}_{23} \setminus \mathsf{eq})$ be an RST relation. Then

$$Reg\mathsf{R} := \{z = x + \imath y \in \mathbb{C} \mid \ ((0,0),(1,0),(x,y)) \in \mathsf{R}\}$$

*is called the **region** of R. z is said to be contained in R. R is called **representable**, and **represented** by the Region Reg R.*

*If $\mathsf{e}_{23} \subset \mathsf{R}$, and $\mathsf{R} \setminus \mathsf{e}_{23}$ is represented by R', then $R' \cup \{\infty\}$ is called the **(Riemann) representation** of R, and R is Riemann representable.*

If the region Reg R of a (Riemann) representable relation R is open, then R is called open.

Note that triples that are in the same RST relation can have different representations. But triples with the same representation are always in the same relation.

A connected representable relation has a connected region.

Example:
As an example, we give the regions of the relations of $\mathcal{LR}$:

$\mathcal{eq}$ is not representable. All others are Riemann representable as follows:

$$\begin{array}{lll} \mathcal{e}_{12} = \{\infty\} & \mathcal{e}_{13} = \{0\} & \mathcal{e}_{23} = \{1\} \\ \mathcal{b} = \,]-\infty, 0[& \mathcal{c} = \,]0,1[& \mathcal{f} = \,]1,\infty[\\ \mathcal{l} = \{z \in \mathbb{C} \mid \Im z > 0\} & \mathcal{r} = \{z \in \mathbb{C} \mid \Im z < 0\} & \end{array}$$

4.3 Mathematical Approach to derive Transformations and Compositions

The following two propositions allow to derive the operations for any RST relations. They are fundamental tools in our central proof in the next section.

Lemma 1 (Transformation lemma).
1. The result of a transformation of an RST relation is again an RST relation.
2. The transformations of Riemann representable relations are Riemann representable relations . For their representations holds:

$$\begin{array}{rcll} z \in Reg\,\mathsf{R} & \iff & (1-z) & \in Reg\,\mathsf{R}^{(12)} \\ & \iff & \frac{1}{z} & \in Reg\,\mathsf{R}^{(23)} \\ & \iff & \frac{z}{z-1} & \in Reg\,\mathsf{R}^{(13)} \\ & \iff & \frac{1}{1-z} & \in Reg\,\mathsf{R}^{(231)} \\ & \iff & 1-\frac{1}{z} & \in Reg\,\mathsf{R}^{(312)} \end{array}$$

Proof:
1. The assertion is trivial for the identity.

Let α be an RST automorphism, R an RST relation. Then:

$$\begin{array}{rcll} (a_{\pi(1)}, \ldots, .a_{\pi(3)}) \in \mathsf{R}^{\pi} & \iff & ((a_1, a_2, a_3)) & \in \mathsf{R} \\ & \iff & (\alpha(a)_1, \ldots, \alpha(a)_3) & \in \mathsf{R} \\ & \iff & (\alpha(a_{\pi(1)}), \ldots, .\alpha(a_{\pi(3)})) & \in \mathsf{R}^{\pi} \end{array}$$

This proves that the RST property is preserved.
2. First assume $z_1 \neq z_2$. Note that

$$\begin{array}{rcl} z = \frac{z_3-z_1}{z_2-z_1} \in Reg\mathsf{R} & \iff & (z_1, z_2, z_3) \in \mathsf{R} \\ & \iff & (z_1^{\pi}, z_2^{\pi}, z_3^{\pi}) := (z_{\pi(1)}, z_{\pi(2)}, z_{\pi(3)}) \in \mathsf{R}^{\pi} \\ & \iff & z^{\pi} := \frac{z_3^{\pi}-z_1^{\pi}}{z_2^{\pi}-z_1^{\pi}} \in Reg\mathsf{R}^{\pi} \end{array}$$

After substituting $z_i^{\pi} = z_{\pi(i)}$ and $z_1 = 0$, $z_2 = 1$, $z_3 = z$, we get the five results for the transformed points z^{π}, e. g.:

$$\begin{array}{l} z_1^{(12)} = z_2 = 1, \\ z_2^{(12)} = z_1 = 0, \\ z_3^{(12)} = z_3 = z, \implies \ z^{(12)} = \frac{z-1}{0-1} = 1-z \end{array}$$

The other formulas are derived analogously. Comparison with the transformations of $\mathcal{E}q$ shows that this arithmetic holds for Riemann representations if we set

$$\frac{1}{0} = \infty,\ \ \infty - 1 = \infty,\ \ 1 - \infty = \infty,\ \ \frac{1}{\infty} = 0 \text{ and } \frac{\infty}{\infty} = 1$$

(qed.)

Example:
This property allows to "calculate" with RST relations. For example, it is possible to determine all $\mathcal{F}$ base relations R for which $\mathsf{R} = \mathsf{R}^{(231)}$ Apart from $\mathcal{e}q$ ($\mathcal{e}q$ is not Riemann representable and must be considered separately), these are exactly the relations { w } for which $w = \frac{1}{1-w}$ holds. There are two solutions:

$$w = \frac{1}{2} + \frac{\sqrt{3}}{2}\imath;\ \bar{w} = \frac{1}{2} + \frac{\sqrt{3}}{2}\imath.$$

Like for transformations, there are arithmetic ways to determine all compositions.

Lemma 2 (Composition lemma).
1. Compositions of RST relations are RST relations.
2. If R_1, R_2 and their transformations are representable, then the composition is continuous, and for the representing regions holds:

$$\begin{aligned}
\mathsf{R}_1 \ {}_3\diamond_2\ \mathsf{R}_2 &= \{(z_1 \cdot z_2) \mid\ z_1 \in Reg(\mathsf{R}_1) \wedge z_2 \in Reg(\mathsf{R}_2)\}\\
\mathsf{R}_1 \ {}_3\diamond_1\ \mathsf{R}_2 &= \{(z_1 + z_2 - z_1 z_2) \mid\ z_1 \in Reg(\mathsf{R}_1) \wedge z_2 \in Reg(\mathsf{R}_2)\}\\
\mathsf{R}_1 \ {}_2\diamond_1\ \mathsf{R}_2 &= \{\tfrac{(z_1 \cdot z_2)}{z_1 + z_2 - 1} \mid\ z_1 \in Reg(\mathsf{R}_1) \wedge z_2 \in Reg(\mathsf{R}_2)\}
\end{aligned}$$

3. *For all* $0 \le \kappa, \lambda \le 3$:

$$\mathsf{R}_1 \ {}_\kappa\diamond_\lambda\ \mathsf{R}_2 = \mathsf{R}_2 \ {}_\kappa\diamond_\lambda\ \mathsf{R}_1$$

$$(\mathsf{R}_1 \ {}_\kappa\diamond_\lambda\ \mathsf{R}_2) \ {}_\kappa\diamond_\lambda\ \mathsf{R}_3 = \mathsf{R}_1 \ {}_\kappa\diamond_\lambda\ (\mathsf{R}_2 \ {}_\kappa\diamond_\lambda\ \mathsf{R}_3)$$

Proof:
For 1., we show that if R_1 and R_2 are RST relations, then $\mathsf{R}_1 \ {}_3\diamond_2\ \mathsf{R}_2$ is an RST relation. Let α be an RST automorphism, then for the RST relations R_i holds:

$$(a_1, a_2, a_3) \in \mathsf{R}_i \iff (\alpha(a_1), \alpha(a_2), \alpha(a_3)) \in \mathsf{R}_i \text{ for } i \in \{1, 2\}.$$

$$\begin{array}{llll}
 & \text{Hence} & (a_1, a_2, a_3) \in \mathsf{R}_1 \ {}_3\diamond_2\ \mathsf{R}_2 & \\
\iff & \exists\, x : & (a_1, a_2, x) \in \mathsf{R}_1 & \text{and } (a_1, x, a_3) \in \mathsf{R}_2 \\
\iff & \exists\, \alpha(x) : & (\alpha(a_1), \alpha(a_2), \alpha(x)) \in \mathsf{R}_1 & \text{and } (\alpha(a_1), \alpha(x), \alpha(a_3)) \in \mathsf{R}_2 \\
\iff & & (\alpha(a_1), \alpha(a_2), \alpha(a_3)) \in (\mathsf{R}_1 \ {}_3\diamond_2\ \mathsf{R}_2) &
\end{array}$$

From the Interdependence of Compositions 1 and the Transformations Lemma (Lemma 1), the property follows for all compositions.

For 2, first we regard the composition $(z_a, z_b, z_x) \in \mathsf{R}_1 \ {}_3\diamond_2\ (z_a, z_x, z_c) \in \mathsf{R}_2$.

If all transformations are representable in $\mathbb{C} \setminus \infty$, then $\frac{1}{z_b - z_a}$, $\frac{1}{z_b - z_c}$ and $\frac{1}{z_d - z_a}$ are defined. Thus

$$
\begin{aligned}
z_3 \in (\mathsf{R}_1 \;{}_3\diamond_2\; \mathsf{R}_2) &\iff \exists\, z_a, z_b, z_c \in \mathbb{C}: \quad z_3 = \tfrac{z_c - z_a}{z_b - z_c},\; (z_a, z_b, z_c) \in \mathsf{R}_1 \;{}_3\diamond_2\; \mathsf{R}_2 \\
&\iff \exists\, z_a, z_b, z_c, z_x \in \mathbb{C}: \; (z_a, z_b, z_x) \in \mathsf{R}_1 \text{ and } (z_a, z_x, z_c) \in \mathsf{R}_2 \\
&\iff \exists\, z_a, z_b, z_c, z_x \in \mathbb{C}: \; \tfrac{z_x - z_a}{z_b - z_a} \in \mathsf{R}_1 \text{ and } \tfrac{z_c - z_a}{z_x - z_a} \in \mathsf{R}_2 \\
&\iff \exists\, z_1, z_2 \in \mathbb{C}: \quad z_1 \in \mathsf{R}_1 \text{ and } z_2 \in \mathsf{R}_2 \text{ and } z_3 = z_1 z_2.
\end{aligned}
$$

The other compositions are calculated with the Conversion Formula 1 and transformation formula 1, e. g.

$$
\begin{aligned}
z_3 \in (\mathsf{R}_1 \;{}_3\diamond_1\; \mathsf{R}_2) &\iff \quad z_3 \in (\mathsf{R}_1^{(12)} \;{}_3\diamond_2\; \mathsf{R}_2^{(12)})^{(12)} \\
&\iff \quad (1 - z_3) \in (\mathsf{R}_1^{(12)} \;{}_3\diamond_2\; \mathsf{R}_2^{(12)}) \\
&\iff \exists\, z_1^{(12)} \in R_1^{(12)}, z_2^{(12)} \in R_2^{(12)} : 1 - z_3 = z_1^{(12)} \cdot z_2^{(12)} \\
&\iff \exists\, z_1 \in R_1, z_2 \in R_2 : \quad 1 - z_3 = (1 - z_1)(1 - z_2) \\
&\iff \exists\, z_1 \in R_1, z_2 \in R_2 : \quad z_3 = z_1 + z_2 - z_1 z_2
\end{aligned}
$$

Because the formulas are symmetric in z_1 and z_2, exchanging z_1 and z_2 does not change the result. Hence $\forall \kappa, \lambda \in \{1,2,3\} : \mathsf{R}_1 \;{}_\kappa\diamond_\lambda\; \mathsf{R}_2 = \mathsf{R}_2 \;{}_\kappa\diamond_\lambda\; \mathsf{R}_1$ Further calculation shows the last equality, e. g.

$$
\begin{aligned}
z_4 \in (\mathsf{R}_1 \;{}_3\diamond_2\; \mathsf{R}_2) \;{}_3\diamond_2\; \mathsf{R}_3 &\iff \exists z_1, z_2, z_3 : z_4 = (z_1 z_2) z_3 = z_1 (z_2 z_3) \\
&\iff z_4 \in \;{}_3\diamond_2\; \mathsf{R}_1 \;{}_\kappa\diamond_\lambda\; (\mathsf{R}_2 \;{}_3\diamond_2\; \mathsf{R}_3); \\
z_4 \in (\mathsf{R}_1 \;{}_3\diamond_1\; \mathsf{R}_2) \;{}_3\diamond_1\; \mathsf{R}_3 &\iff \exists z_1, z_2, z_3 : \\
&\qquad z_4 = (z_1 + z_2 - z_1 z_2) + z_3 - (z_1 + z_2 - z_1 z_2) z_3 \\
&\qquad\quad = z_1 z_2 z_3 - z_1 z_2 - z_2 z_3 - z_3 z_1 + z_1 + z_2 + z_3 \\
&\iff z_4 \in \;{}_3\diamond_1\; \mathsf{R}_1 \;{}_\kappa\diamond_\lambda\; (\mathsf{R}_2 \;{}_3\diamond_1\; \mathsf{R}_3)
\end{aligned}
$$

All these mappings are continuous. The other compositions are concatenations of the composition ${}_3\diamond_2$ and transformation operations, hence continuous functions.

(qed.)

As we see, compositions and transformations of $\mathcal{F}$ are represented by sets of complex numbers, hence relations in $\mathcal{F}$. This means, $\mathcal{F}$ is closed under composition and transformation. Although being fundamental for the definition of compositions of other RST calculi, $\mathcal{F}$ is not practical because $\mathcal{F}$ has infinitely many relations (For any complex number, there is a base relation of $\mathcal{F}$). We want to find a practical, i. e. finite coarser calculus that is closed under composition and transformation operation.

Is our standard example, $\mathcal{LR}$, closed under composition? From Propositions 1 and 2, we know that it is sufficient to regard one composition, e. g. ${}_3\diamond_2$, on the base relations. With the theory of RST relations, the complete composition table of $\mathcal{LR}$ can be generated. There are two cases: Either one of the base relations is $\mathfrak{eq}$, $\mathfrak{e}_{12}$ or $\mathfrak{e}_{13}$. Then the resulting constraints on the equality of points needs to be checked.

All other compositions, mostly involving refinements of $\mathfrak{d} = \mathfrak{b} \cup \mathfrak{c} \cup \mathfrak{f} \cup \mathfrak{l} \cup \mathfrak{l}$, can be calculated using the composition lemma. Table 3 lists the compositions of ${}_3\diamond_2$. This shows that $\mathcal{LR}$ is practical.

${}_3\diamond_2$	eq	e_{12}	e_{13}	e_{23}	b	c	f	l	r
eq	eq	e_{12}	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
e_{12}	e_{12}	$\emptyset$	eq	e_{12}	e_{12}	e_{12}	e_{12}	e_{12}	e_{12}
e_{13}	e_{13}	$d \cup e_{23}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
e_{23}	$\emptyset$	$\emptyset$	e_{13}	e_{23}	b	c	f	l	r
b	$\emptyset$	$\emptyset$	e_{13}	b	$c \cup e_{23} \cup f$	b	b	r	l
c	$\emptyset$	$\emptyset$	e_{13}	c	b	c	$c \cup e_{23} \cup f$	l	r
f	$\emptyset$	$\emptyset$	e_{13}	f	b	$c \cup e_{23} \cup f$	f	l	r
l	$\emptyset$	$\emptyset$	e_{13}	l	r	l	l	$l \cup r \cup b$	$d \cup e_{23}$
r	$\emptyset$	$\emptyset$	e_{13}	r	l	r	r	$d \cup e_{23}$	$l \cup r \cup b$

Table 3. Compositions of $\mathcal{LR}$. Note that d is an abbreviation for $b \cup c \cup f \cup l \cup r$

5 The special role of $\mathcal{LR}$

When studying point-based qualitative representation calculi, one notices that they all share a particular property. For all tuples in one base relation, we obtain connected regions when we vary one point and leave all other points constant. In fact, it would appear to be very "unnatural" if one would get unconnected regions. Such sets of unconnected regions one would only expect if a relation is truly disjunctive. Since this property turns out to be very important, we give it a name in the next definition.

Definition 8 (Natural calculus) *An RST calculus is called **natural** if for all its representable base relations* B, $Reg\mathsf{B}$ *and* $\mathbb{C} \setminus Reg\mathsf{B}$ *are connected.*

Example:
In $\mathcal{LR}$, all base relations and the complements of base relations are connected.

In this section, we will prove that $\mathcal{LR}$ is the finest practical natural RST calculus.

5.1 Some definitions and lemmata

First, we need some more definitions and lemmata to structure the proof.

Definition 9 (Bounded relation)
A representable relation R *except* $\{0\}, \{1\}$ *is called bounded if* $Reg(\mathsf{R})$ *is bounded.*

The exceptions $\{0\}, \{1\}$ are omitted because not all of their transformations are representable.

Now the proof will be structured as follows: First we prove some properties of bounded and other representable relations, then we will use them to derive limitations for bounded relations in practical calculi. Then we prove for all $\mathcal{LR}$ relations that no connected refinement of $\mathcal{LR}$ relations will fulfill these criteria.

Lemma 3 (Bounded composition and inverse).
For representable relations holds:

1. If R *is bounded, then* $\mathsf{R}^{(12)}$ *is bounded.*

2. If R_1 and R_2 are bounded, then the composition $\mathsf{R} = \mathsf{R}_1 \; {}_3\diamond_2 \; \mathsf{R}_2$ is bounded, or $\mathsf{R}=\{1\}$

Proof:
1.This follows directly from Composition Lemma 2 and Transformation Lemma 1. Let $z \in Reg\mathsf{R}$, then $1 - z \in Reg(\mathsf{R}^{(12)})$. Let S be the upper boundary for $Reg(\mathsf{R})$, hence $|z| < S$ for all z in R, then $S + 1$ is an upper boundary for $|1 - z|$. The exceptions, $\{0\}, \{1\}$ cannot be the resulting relation because $\{0\}^{(12)} = \{1\}$, and $\{1\}^{(12)} = \{0\}$, but R is neither $\{0\}$ nor $\{1\}$.

2. $\mathsf{R}_i \neq \{0\} \Rightarrow \exists z_i \neq 0 : \; z_i \in R_i \Rightarrow z := z_1 \cdot z_2 \in R, z \neq 0$. Hence $\mathsf{R} \neq \{0\}$. With the Composition Lemma 2 we obtain: If S_i is upper boundary for $Reg(\mathsf{R}_i)$, then $S_1 \cdot S_2$ is upper boundary for the Composition, because $z \in \mathsf{R}_1 \; {}_3\diamond_2 \; \mathsf{R}_2$ iff $z = z_1 \cdot z_2$ (with $|z_i| \leq S_i$). Then $|z| \leq S_1 \cdot S_2$. The exception $\mathsf{R}_1 \; {}_3\diamond_2 \; \mathsf{R}_2 = \{0\}$ cannot occur because both relations contain values $z_1, z_2 \neq 0$, thus $z = z_1 \cdot z_2 \in \mathsf{R}_1 \; {}_3\diamond_2 \; \mathsf{R}_2, z \neq 0$. (qed.)

Lemma 4 (0-1-lemma).
Let $\mathcal{C}$ be a practical RST calculus. Then :

1. For each relation $\mathsf{R} \not\subseteq \{eq, e_{12}, b, e_{13}\}$ *in* $\mathcal{C}$*:* $1 \in \partial(Reg\mathsf{R})$

2. If R is bounded, then $\limsup_{z \in \mathsf{R}} |z| = 1$.

3. If R is bounded, then $Reg(\mathsf{R}) \subseteq \mathbb{R}^+$

4. If R is bounded, then $0 \in \partial(Reg\mathsf{R})$.

This shows that only some bounded relations occur in practical calculi. As we will see later, refining $\mathcal{LR}$ would require bounded regions that do not fulfill the criteria of the 0-1-Lemma.

Proof:
First we will prove properties 2 - 4, and then derive property 1 for possibly unbounded relations at the end of this proof, as a generalization of property 4 for bounded relations. We start with the proof of property 2.

Let R be bounded and $S = \limsup_{z \in \mathsf{R}} |z|$ be the smallest upper limit. We will show that S=1. Due to the Composition Lemma 2 we know that

$$\mathsf{R}^k := \mathsf{R} \; {}_3\diamond_2 \; \mathsf{R} \ldots \; {}_3\diamond_2 \; \mathsf{R} = \{z \mid \exists z_1, \ldots, z_k \in \mathsf{R} : z = z_1 z_2 \ldots z_k\}.$$

Multiplication is a continuous and monotonous function, hence

$$\limsup_{z \in \mathsf{R}^k} |z| = S^k$$

Suppose $S \neq 1$, then all these infinitely many relations $(\mathsf{R}^k)_{k \in \mathbb{N}}$ were pairwise different, and all of them were contained in $\mathcal{C}$ because $\mathcal{C}$ is closed under compositions. But $\mathcal{C}$ is finite, hence $S = 1$. This proves property 2.

For 3., suppose, $|z| = r(cos\phi + \imath\, sin\phi)$ $(\phi \neq 0)$ is contained in R. Then R^k contains $z^k = r^k(\cos(k\phi) + \imath\ \sin(k\phi))$. For some k, $\Re(z^k) < 0$. Due to the previous Lemma 3 the relation $(\mathsf{R}^k)^{(12)}$ is bounded. By the Transformation Lemma 1, $(\mathsf{R}^k)^{(12)}$ contains $1 - z^k$ with $|1 - z^k| \geq \Re(1 - z^k) > 1$ in contradiction to 2. This proves property 3. (Refer to Figure 4)

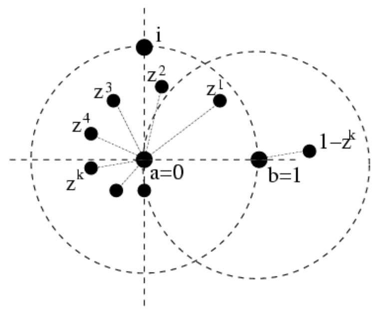

Fig. 4. Sketch for the proof of the 0-1-lemma. For some k, we have $1 - z^k \in (\mathsf{R}^k)^{(12)}$ and $|1 - z^k| > 1$.

For the proof of 4., suppose, $0 \notin \partial\ \mathsf{R}$. Then $\mathsf{R}^{(23)}$ is bounded because $\mathsf{R}^{(23)} = \{z \mid \frac{1}{z} \in \mathsf{R}\}$ (Transformation Lemma 1). For $\mathsf{R}^{(23)}$ in $\mathcal{C}$, we have

$$\limsup_{z \in \mathsf{R}^{(23)}} |z| = 1, \text{ hence } \liminf_{z \in \mathsf{R}} |z| = 1.$$

Thus for all $z \in \mathsf{R} : |z| = 1$. From property 3. follows $\mathsf{R} = \{1\}$, but by definition, $\{\ 1\ \}$ is not bounded. Hence the supposition is wrong, which proves 4.

Property 1 for possibly unbounded relations can be derived from property 4. It is sufficient to prove the claim in $1 \in \partial(\mathsf{R} \setminus \{1\})$. Let $\mathsf{R}\backslash\{1\}$ be an arbitrary relation $\mathsf{R} \not\subseteq \{\mathit{eq}, \mathit{e}_{12}, \mathit{b}, \mathit{e}_{13}\}$. Suppose, $1 \notin \partial\ \mathsf{R}$. Then $0 \notin \partial \mathsf{R}^{(12)}$. By the Transformation Lemma 1 $(\mathsf{R}^{(12)})^{(23)}$ is bounded. From 3. and 4., we know $(\mathsf{R}^{(12)})^{(23)} \subseteq [0, 1]$. Hence $\mathsf{R}^{(12)} \subseteq [1, \infty]$ and $\mathsf{R}^{(12)} \subseteq [-\infty, 0] = \mathit{b} \cup \mathit{e}_{13}$, in contradiction to the prerequisite. This completes the proof. (qed.)

The restrictions in the 0-1-lemma apply mainly because a practical calculus has to be finite. As a consequence, the demand for a finite calculus limits the number of refinements of RST calculi.

5.2 The Central Theorem

Theorem 1 (Central Theorem).
Any practical natural refinement of $\mathcal{LR}$ is $\mathcal{LR}$ itself.

The proof is based on the fact that the relations of practical calculi fulfill the properties of the 0-1-Lemma 4. A close consequence is the following lemma that shows that the relation c cannot be further refined.

Lemma 5 (connected bounded relations).
If $\mathcal{C}$ is a practical *calculus. Then any connected bounded relation in $\mathcal{C}$ is a whole interval from 0 to 1.*

Proof:
Let R be a connected bounded relation in $\mathcal{C}$. According to the 0-1-lemma, a bounded relation is a subset of the interval $[0,1]$. Suppose there is some $x \in \ ]0,1[\ , x \notin Reg\,\mathsf{R}$. Since $Reg\,\mathsf{R}$ is connected, all points are below or above x:

$$\limsup_{z\in\mathsf{R}} |z| \le x \qquad \text{or} \qquad \liminf_{z\in\mathsf{R}} |z| \ge x$$
$$\text{and then} \ \limsup_{z\in\mathsf{R}^{(12)}} |z| \le 1-x.$$

In both cases, this contradicts the 0-1-lemma. Hence $]0,1[\ \subset Reg\,\mathsf{R}$. (qed.)

$Reg(\mathsf{c}) = \ ]0,1[$. Hence neither the relation c nor its transformations b and f can be further refined. eq and the point relations e_{12}, e_{13}, e_{23} cannot be refined because they are already relations of $\mathcal{F}$. The task is now to prove that l and r cannot be refined: Unlike with f and b, we cannot refer to bounded transformations because the transformations of l and r are l and r. However, using criteria 1. from the 0-1-lemma, we can show: If there is a refinement and all base relations are connected, then there is a bounded subrelation of l, which contradicts the previous Lemma 5.

In practical calculi, for all base relations B, either

$$B \cap B^{(231)} \cap B^{(321)} = \emptyset$$
$$\text{or} \ \ B \cap B^{(231)} \cap B^{(321)} = B,$$

holds because the intersection is a relation of the calculus. We will call base relations for which the second line is true, ***rotation-symmetric***, because they remain unchanged under the "rotating" transformations. (Note that $B^{(231)}$ and $B^{(321)}$ are base relations, hence $B^{(231)} = B^{(321)} = B$).

To simplify the proof, we transform the plane of complex numbers so that the "rotating" transformations actually are represented by a rotation of the complex plane with center 0. This allows us to use the inherent symmetry in the proof.

Therefore, regard the Mobius transformation μ, that maps the half plane $Reg(\mathsf{l}) = \{z \mid \Im(z) > 0\}$ to the unit disk :

$$\mu(z) := \frac{z-w}{z-\bar{w}} \text{ , where } w = \sqrt[3]{-1} = \frac{1}{2} + \frac{\sqrt{3}}{2}\imath$$

Remark 1 (Rotating Transformations)

$$\mu(0) = w^2,\ \mu(1) = w^4,\ \mu(\infty) = 1,\ \mu(w) = 0$$

$$\mu(\mathsf{R}^{(231)}) = w^2 \cdot \mu(\mathsf{R}), \quad \mu(\mathsf{R}^{(321)}) = w^4 \cdot \mu(\mathsf{R}),$$

hence the images of rotation-symmetric base relations are rotation-symmetric with angle 120^o *around* $\mu(w) = 0$.

Proof:
First we state some equalities: With $w^6 = 1,\ w^5 = \bar{w},\ \bar{w}w = 1 = \bar{w} + w$, we obtain

$$\frac{\frac{1}{1-z} - w}{\frac{1}{1-z} - \bar{w}} = \frac{1-w+wz}{1-\bar{w}+\bar{w}z} = \frac{\bar{w}+wz}{w+\bar{w}z} = \frac{1+w^2z}{w^2+z} = w^2\left(\frac{z-w}{z-\bar{w}}\right)$$
$$\text{and } \frac{1-\frac{1}{z}-w}{1-\frac{1}{z}-\bar{w}} = \frac{z-1-wz}{z-1-\bar{w}z} = \frac{(1-w)z-1}{1-\bar{w}z-1} = \frac{\bar{w}z-1}{wz-1} = \frac{w^5z-w^6}{wz-w\bar{w}} = w^4\left(\frac{z-w}{z-\bar{w}}\right)$$

The Transformation Lemma now completes the proof:

$$\begin{array}{rcccc} z \in Reg\mathsf{R} & \Longleftrightarrow & 1-\frac{1}{z} \in Reg\mathsf{R}^{(321)} & \Longleftrightarrow & \frac{1}{1-z} \in Reg\mathsf{R}^{(231)} \\ \Rightarrow \frac{z-w}{z-\bar{w}} \in \mu(Reg\mathsf{R}) & \Longleftrightarrow & \frac{\frac{1}{1-z}-w}{\frac{1}{1-z}-\bar{w}} \in \mu(Reg\mathsf{R}^{(231)}) & \Longleftrightarrow & w^2\left(\frac{z-w}{z-\bar{w}}\right) \in \mu(Reg\mathsf{R}^{(231)}), \\ \frac{z-w}{z-\bar{w}} \in \mu(Reg\mathsf{R}) & \Longleftrightarrow & \frac{1-\frac{1}{z}-w}{1-\frac{1}{z}-\bar{w}} \in \mu(Reg\mathsf{R}^{(321)}) & \Longleftrightarrow & w^4\left(\frac{z-w}{z-\bar{w}}\right) \in \mu(Reg\mathsf{R}^{(321)}) \end{array}$$

(qed.)

Note that Mobius transformations preserve connectivity. If $\mu(Reg\mathsf{R})$ is connected, then R is connected.

Lemma 6 (Rotation symmetry).
For a practical calculus with connected base relations holds: A base relation B *is rotation-symmetric iff* $w \in \mathsf{B}$ *or* $\bar{w} = w^5 \in \mathsf{B}$.

Proof:
If $w \in Reg\mathsf{B}$, $\mu(w) = 0$, follows $w \in Reg\mathsf{B}^{(231)}$ by the Remark on Rotating Transformations. If $\bar{w} \in Reg\mathsf{B}$, then $1 - \frac{1}{\bar{w}} = \overline{(1-\frac{1}{w})} = \bar{w} \in Reg\mathsf{B}^{12}$, hence in both cases $\mathsf{B} \cap \mathsf{B}^{(12)} \neq \emptyset$. Thus $\mathsf{B} = (\mathsf{B})^{(231)}$ is rotation-symmetric .

For the reverse direction, we use the following consequence of the Jordan Curve Theorem ([18]): If v is a continuous closed path in $\mathbb{C} \setminus \{\infty\}$ that does not contain w but does go around a point z at least once, then the set of all points that can be reached from z by a continuous path that does not intersect v is bounded. (For the proof refer to [15].)

Suppose that there is a representable rotation-symmetric base relation B that contains neither w nor $\bar{w}$. Then because of connectivity, there is a continuous path v from a point $z \in \mu(Reg\mathsf{B})$ to $w^2z \in \mu(Reg(\mathsf{B}^{(231)}))$. Then $w^2 \cdot v$ is a continuous path from w^2z to w^4z, and $w^4 \cdot v$ a continuous path from w^4z to $w^6z = z$. Because of the rotation symmetry of $\mathsf{B} = \mathsf{B}^{(231)}$, all these paths remain in $\mu(Reg\mathsf{B})$, and together they form a closed path that goes around 0. By assumption, this path does not hit but separates $0 = \mu(w)$ and $\infty = \mu(\bar{w})$, because $w, \bar{w} \notin \mathsf{B}$. Either $\mu(w) = 0$ and $\mu(\infty) = 1$ are on different sides of the path, then the base relation B_w, containing w is bounded, or $\mu(\bar{w}) = \infty$ and $\mu(\infty) = 1$ are on different sides of the path, then the base relation $\mathsf{B}_{\bar{w}}$ containing $\bar{w}$ is bounded. Here, we use the connectivity of the base relations.

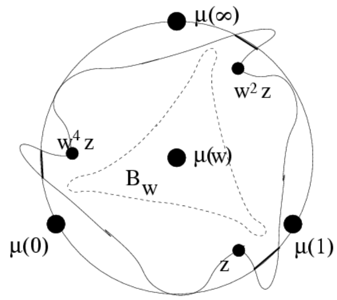

Fig. 5. Sketch for the proof of the Central Theorem.

This contradicts the 0-1-lemma. (qed.)

Now we complete the proof of the Central Theorem. We have to show that l and r cannot properly be refined.

Let us assume there is a proper refinement of l in a natural calculus, hence exits $\mathsf{B}_w \subsetneq \mathsf{l}$ with $w \in Reg\mathsf{B}_w$. Let $\mathsf{C}_w = \top \setminus \mathsf{B}_w$. In a natural calculus, C_w is connected. In a proper refinement, $\mathsf{C}_w \cap \mathsf{l} \neq \emptyset$, hence there is $z \in \mathsf{C}_w \cap \mathsf{l}$. Note that $\Im(z) > 0$, hence $|\mu(z)| < 1$. $\mathsf{C}_w = \mathsf{C}_w^{(231)} = \mathsf{C}_w^{(321)}$ as the complement of the rotation-symmetric relation B_w. Then $w^2\mu(z), w^4\mu(z) \in \mu(Reg(\mathsf{C}_w)$, and there is a continuous closed path v from $\mu(z)$ to $w^2\mu(z)$ to $w^4\mu(z)$ to $\mu(z)$ within $\mu(Reg\mathsf{C})$. Without loss of generality, we assume that this path remains inside the closed unit disk $\mu(\{z \mid \Im(z) \geq 0\})$ because $\mathbb{R} \subset Reg(\mathsf{C}_w)$ (refer to figure 5). Note that $0 = \mu(w)$ is in the inside of the path. Because B_w is connected, any point of $\mu(\mathsf{B}_w)$ is inside this path, hence B_w is bounded but $Reg\mathsf{B}_w \not\subset \mathbb{R}$, in contradiction to the 0-1-Lemma 4. This means, the assumption is wrong - l has no proper refinement in practical natural calculi. By reasons of symmetry, the same holds for r. This proves the Central Theorem 1. (qed.)

6 Conclusion

We developed a general theory for ternary point-based calculi such that the relations are invariant when all points are mapped by rotations, scalings or translations. These calculi are called RST calculi. Examples for such RST calculi are Freksa's double cross calculus [3, 4], Ligozat's flip-flop calculus [8], and Moratz et al.'s TPCC calculus [11].

We argued that one requirement on an RST calculus is that it should be "practical," i.e., that it is closed under the usual operations and that it possesses a finite JEPD base. This is a prerequisite for applying Montanari's path-consistency algorithm [10] in a way such that no information loss occurs. Further, we required an RST calculus to be "natural," which means that the region denoted by a relation is internally connected.

The original versions of the double-cross and the flip-flop calculus fail the practicality test since their relations are not jointly exhaustive because some equality relations are missing. After adding these relations one arrives at the following results. As shown elsewhere [16], the double-cross calculus does not have a finite base, while on the other hand, the completion of the much coarser flip-flop calculus, which we named $\mathcal{LR}$, is a practical and natural calculus. As the main result of the paper, we were able to show that $\mathcal{LR}$ cannot be properly refined without loosing this property. From that it follows, e.g., that Moratz et al.'s TPCC [11] does not have a finite base, which was unknown until now.

An interesting direction of further research could be to make use of the method to compute exact compositions and transformations in the infinite finest RST calculus $\mathcal{F}$ presented in this paper. This might provide a new solution to conclude knowledge in some cases, even with infinite calculi.

References

1. M. J. Egenhofer. Reasoning about binary topological relations. In O. Günther and H.-J. Schek, editors, *Proceedings of the Second Symposium on Large Spatial Databases, SSD'91*, volume 525 of *Lecture Notes in Computer Science*, pages 143–160. Springer-Verlag, Berlin, Heidelberg, New York, 1991.
2. A. Frank. Qualitative spatial reasoning with cardinal directions. In *Proceedings of the Seventh Austrian Conference on Artificial Intelligence*, Berlin, Heidelberg, New York, 1991. Springer-Verlag.
3. C. Freksa. Using orientation information for qualitative spatial reasoning. In A. U. Frank, I. Campari, and U. Formentini, editors, *Theories and Methods of Spatio-Temporal Reasoning in Geographic Space*, pages 162–178. Springer-Verlag, Berlin, Heidelberg, New York, 1992.
4. C. Freksa and K. Zimmermann. On the utilization of spatial structures for cognitively plausible and efficient reasoning. In *Proc. of the IEEE International Conference on Systems, Man, and Cybernetics*, pages 261–266, Chicago, IL, 1992. IEEE.
5. A. Isli and A. G. Cohn. A new approach to cyclic ordering of 2D orientations using ternary relation algebras. *Artificial Intelligence*, 122(1-2):137–187, 2000.
6. A. Isli and R. Moratz. Qualitative spatial representation: Algebraic models for relative position. Technical Report FBI-HH-M284/99, Univ. Hamburg, 1999.
7. P. B. Ladkin and R. Maddux. On binary constraint problems. *Journal of the Association for Computing Machinery*, 41(3):435–469, May 1994.
8. G. Ligozat. Qualitative triangulation for spatial reasoning. In *Spatial Information Theory: A Theoretical Basis for GIS (COSIT-93)*, pages 54–68, 1993.
9. N. Megiddo. Partial and complete cyclic orders. *Bull. Am. Math. Soc*, 82:274–276, 1976.
10. U. Montanari. Networks of constraints: fundamental properties and applications to picture processing. *Information Science*, 7:95–132, 1974.
11. R. Moratz, B. Nebel, and C. Freksa. Qualitative spatial reasoning about relative position. In F. et al., editor, *Spatial Cognition III*, pages 385–400. Springer-Verlag, 2003.
12. D. Papadias, M. J. Egenhofer, and J. Sharma. Hierarchical reasoning about direction relations. In *GIS '96, Proceedings of the fourth ACM workshop on Advances on Advances in Geographic Information Systems, November 15-16, 1996, Rockville, Maryland, USA*, pages 105–112, 1996.
13. D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In B. Nebel, W. Swartout, and C. Rich, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the 3rd International Conference (KR-92)*, pages 165–176, Cambridge, MA, Oct. 1992. Morgan Kaufmann.

14. C. Schlieder. Reasoning about ordering. In A. U. frank and W. Kuhn, editors, *Spatial Information Theory: A Theoretical Basis for GIS (COSIT-95)*, pages 341–349, 1995.
15. A. Scivos. Einführung in eine Theorie der ternären RST-Kalküle für qualitatives räumliches Schließen. Diplomarbeit, Albert-Ludwigs-Universität Freiburg, Mathematische Fakultät, 2000.
16. A. Scivos and B. Nebel. Double-crossing: Decidability and computational complexity of a qualitative calculus for navigation. In *Spatial Information Theory: Foundations of Geographic Information Science (COSIT-2001)*, Morro Bay, CA, 2001. Springer-Verlag.
17. A. Tarski. On the calculus of relations. *Journal of Symbolic Logic*, 6:73–89, 1941.
18. T. tom Dieck. *Topologie*. de Gruyter, Berlin, 2000.