

Qualitative Representation and Reasoning

Allen's Interval Calculus

Knowledge Representation and Reasoning

January 18, 2006

Allen's Interval Calculus – Outline

Allen's Interval Calculus

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Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- ▶ **NLP** – we do not have precise time points
- ▶ **Planning** – we do not want to commit to time points too early
- ▶ **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- ▶ **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- ▶ **Time intervals**: actions and events have duration
- ▶ Reducibility? Expressiveness? Computational costs for reasoning?

Motivation: Example

Consider a planning scenario for multimedia generation:

P1: *Display* Picture1

P2: Say “Put the plug in.”

P3: Say “The device should be shut off.”

P4: *Point to* Plug-in-Picture1.

Temporal relations between events:

P2	should happen during	P1
P3	should happen during	P1
P2	should happen before or directly precede	P3
P4	should happen during or end together with	P2

~> P4 happens before or directly precedes P3”

~> We could add the statement “P4 does not overlap with P3” without creating an inconsistency.

Allen's Interval Calculus

- ▶ Allen's interval calculus: **time intervals** and **binary relations** over them
- ▶ **Time intervals**: $X = (X^-, X^+)$, where X^- and X^+ are interpreted over the reals and $X^- < X^+$ (\rightsquigarrow naïve approach)

- ▶ **Relations** between concrete intervals, e. g.:

(1.0,2.0) *strictly before* (3.0,5.5)

(1.0,3.0) *meets* (3.0,5.5)

(1.0,4.0) *overlaps* (3.0,5.5)

...

\rightsquigarrow Which relations are conceivable?

The Base Relations

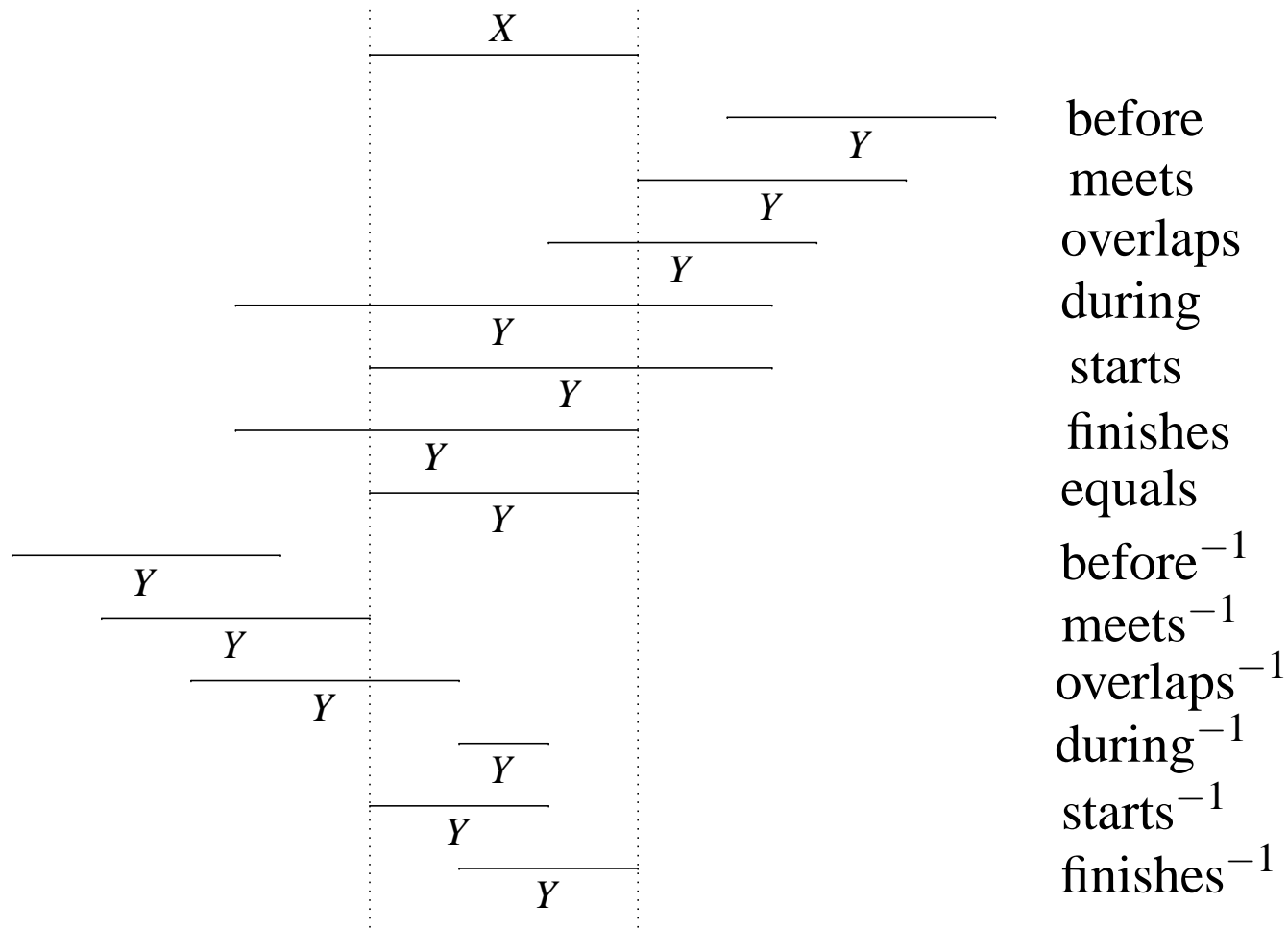
How many ways are there to order the four points of two intervals?

Relation	Symbol	Name
$\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$	\prec	before
$\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$	m	meets
$\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$	o	overlaps
$\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$	s	starts
$\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$	f	finishes
$\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$	d	during
$\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$	\equiv	equal

and the **converse** relations (obtained by exchanging X and Y)

↪ These relations are JEPD.

The 13 Base Relations Graphically



Disjunctive Descriptions

- ▶ Assumption: We don't have precise information about the relation between X and Y , e. g.:

$$X \circ Y \text{ or } X \text{ m } Y$$

- ▶ ... modelled by sets of base relations (meaning the union of the relations):

$$X \{ \circ, \text{m} \} Y$$

↪ 2^{13} imprecise relations (incl. \emptyset and \mathbf{B})

Example of an indefinite qualitative description:

$$\left\{ X \{ \circ, \text{m} \} Y, Y \{ \text{m} \} Z, X \{ \circ, \text{m} \} Z \right\}$$

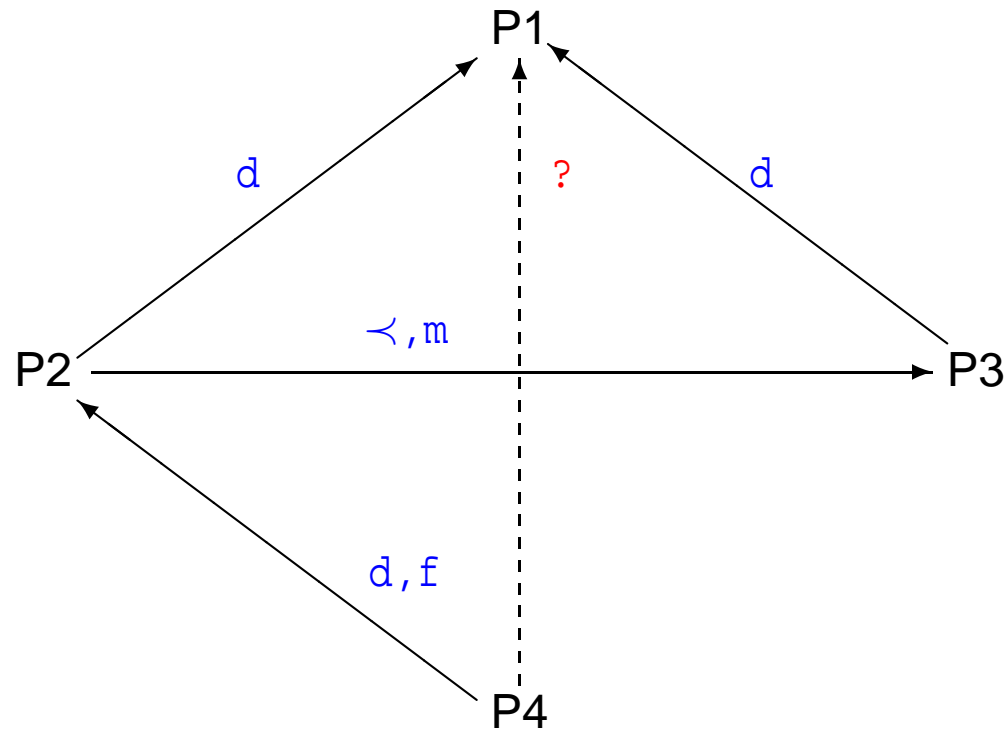
Our Example ... Formal

P1: *Display* Picture1

P2: Say "Put the plug in."

P3: Say "The device should be shut off."

P4: *Point to* Plug-in-Picture1.



Compose the constraints: $P4 \{d, f\} P2$ and $P2 \{d\} P1$: **$P4 \{d\} P1$.**

	λ	γ	d	d^{-1}	o	o^{-1}	m	m^{-1}	s	s^{-1}	f	f^{-1}
λ	λ	B	λo md s	λ	λ	λo md s	λ	λo md s	λ	λ	λo md s	λ
γ	B	γ	γo^{-1} $m^{-1}d$ f	γ	γo^{-1} $m^{-1}d$ f	γ	γo^{-1} $m^{-1}d$ f	γ	γo^{-1} $m^{-1}d$ f	γ	γ	γ
d	λ	γ	d	B	λo md s	γo^{-1} $m^{-1}d$ f	λ	γ	d	γo^{-1} $m^{-1}d$ f	d	λo md s
d^{-1}	λo f^{-1}	γo^{-1} $m^{-1}d^{-1}$ s^{-1}	B $\lambda \gamma$ mm^{-1}	d^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	o^{-1} d^{-1} s^{-1}	o d^{-1} f^{-1}	d^{-1}	o^{-1} d^{-1} s^{-1}	d^{-1}
o	λ	γo^{-1} $m^{-1}d^{-1}$ s^{-1}	o d s	λo md^{-1} f^{-1}	λ o m	B $\lambda \gamma$ mm^{-1}	λ	o^{-1} d^{-1} s^{-1}	o	d^{-1} f^{-1} o	d s o	λ o m
o^{-1}	λo f^{-1}	γ	o^{-1} d f	γ, o^{-1} $m^{-1}d^{-1}$ s^{-1}	B $\lambda \gamma$ mm^{-1}	γ o^{-1} m^{-1}	o d^{-1} f^{-1}	γ	o^{-1} d f	o^{-1} γ m^{-1}	o^{-1} d^{-1} s^{-1}	o^{-1} d^{-1} s^{-1}
m	λ	γo^{-1} $m^{-1}d^{-1}$ s^{-1}	o d s	λ	λ	o d s	λ	f f^{-1} \equiv	m	m	d s o	λ
m^{-1}	λo f^{-1}	γ	o^{-1} d f	γ	o^{-1} d f	γ	s s^{-1} \equiv	γ	d f o^{-1}	γ	m^{-1}	m^{-1}
s	λ	γ	d	λo md^{-1} f^{-1}	λ o m	o^{-1} d f	λ	m^{-1}	s	s s^{-1} \equiv	d	λ m o
s^{-1}	λo f^{-1}	γ	o^{-1} d f	d^{-1}	o d^{-1} f^{-1}	o^{-1}	o d^{-1} f^{-1}	m^{-1}	s s^{-1} \equiv	s^{-1}	o^{-1}	d^{-1}
f	λ	γ	d	γo^{-1} $m^{-1}d^{-1}$ s^{-1}	o d s	o^{-1} γ m^{-1}	m	γ	d	o^{-1} γ m^{-1}	f	f f^{-1} \equiv
f^{-1}	λ	γo^{-1} $m^{-1}d^{-1}$ s^{-1}	o d s	d^{-1}	o	o^{-1} d^{-1} s^{-1}	m	s^{-1} o^{-1} d^{-1}	o	d^{-1}	f f^{-1} \equiv	f^{-1}

Outlook

- ▶ Using the **composition table** and the rules about operations on relations, we can **deduce** new relations between time intervals.
- ▶ What would be a **systematic** approach?
- ▶ How costly is that?
- ▶ Is that **complete**?
- ▶ If not, could it be complete on a subset of the relation system?

Reasoning in Allen's Interval Calculus

Allen's Interval Calculus

Reasoning in Allen's Interval Calculus

Constraint propagation algorithms (enforcing path consistency)

Example for Incompleteness

NP-Hardness Example

The Continuous Endpoint Class

Completeness for the CEP Class

A Maximal Tractable Sub-Algebra

Literature

Constraint Propagation – The Naive Algorithm

Enforcing path-consistency using the straight-forward method:

Let $Table[i, j]$ be an array of size $|n| \times |n|$ (n : number of intervals), in which we have recorded the constraints between the intervals.

EnforcePathConsistency1 (C):

Input: a (binary) CSP $C = \langle V, D, C \rangle$

Output: an equivalent, but path consistent CSP C'

repeat

for each pair (i, j) , $1 \leq i, j \leq n$

for each k with $1 \leq k \leq n$

$Table[i, j] := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

endfor

endfor

until no entry in $Table$ is changed

↪ terminates;

↪ needs $O(n^5)$ intersections and compositions.

An $O(n^3)$ Algorithm

EnforcePathConsistency2 (\mathcal{C}):

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$

Output: an equivalent, but path consistent CSP \mathcal{C}'

$Paths(i, j) = \{(i, j, k) : 1 \leq k \leq n\} \cup \{(k, i, j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i,j} Paths(i, j)$

While $Q \neq \emptyset$

 select and delete (i, k, j) from Q

$T := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

if $T \neq Table[i, j]$

$Table[i, j] := T$

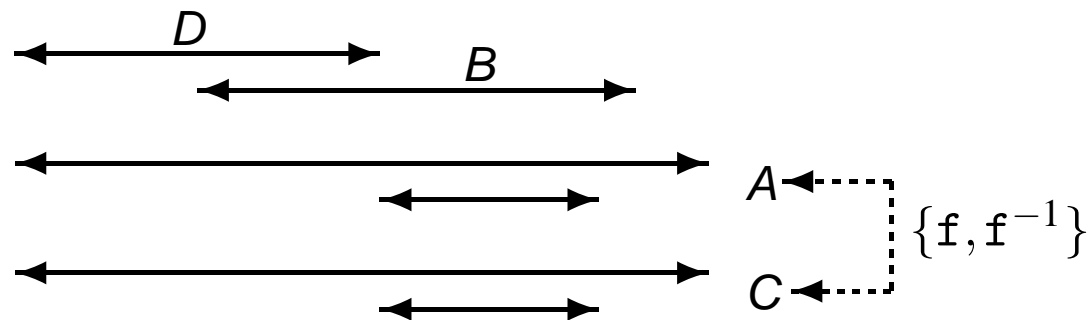
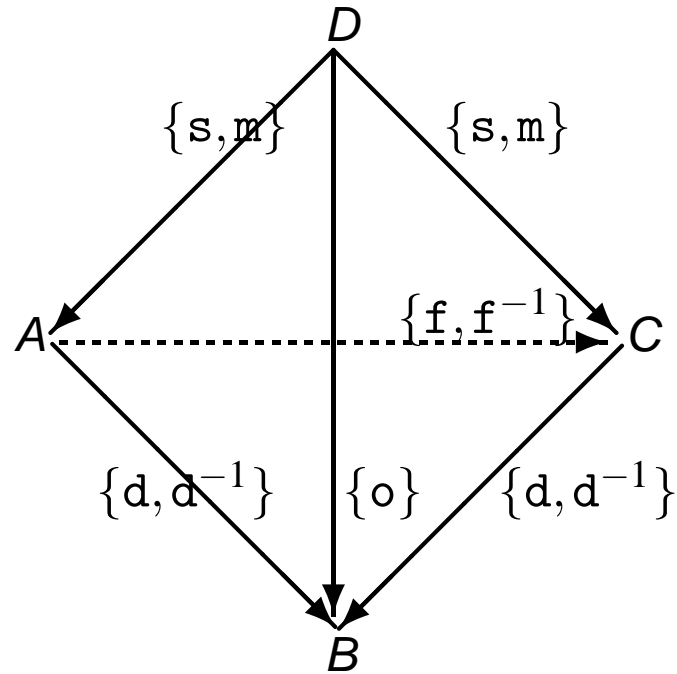
$Table[j, i] := T^{-1}$

$Queue := Queue \cup Paths(i, j)$

endif

endwhile

Example for Incompleteness



NP-Hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.

Reduction from **3-colorability** (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \dots, v_n\}$ be an instance of 3-colorability.

Then we use the intervals $\{v_1, \dots, v_n, 1, 2, 3\}$ with the following constraints:

$$\begin{array}{llll}
 1 & \{m\} & 2 & \\
 2 & \{m\} & 3 & \\
 v_i & \{m, \equiv, m^{-1}\} & 2 & \forall v_i \in V \\
 v_i & \{m, m^{-1}, \prec, \succ\} & v_j & \forall (v_i, v_j) \in E
 \end{array}$$

This constraint system is satisfiable *iff* G can be colored with 3 colors. □

Looking for Special Cases

- ▶ *Idea*: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- ▶ *Note*: Interval formulae $X R Y$ can be expressed as **clauses** over **atoms** of the form $a op b$, where:
 - ▶ a and b are endpoints X^-, X^+, Y^- and Y^+ and
 - ▶ $op \in \{<, >, =, \leq, \geq\}$.
- ▶ *Example*: All base relations can be expressed as unit clauses.

Lemma

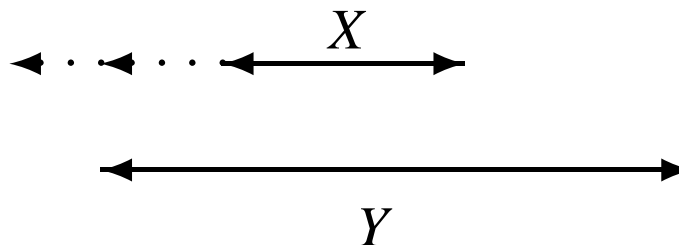
Let $\pi(\Theta)$ be the translation of Θ to clause form. Θ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

The Continuous Endpoint Class

Continuous Endpoint Class \mathcal{C} : This is a subset of \mathcal{A} such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \{ X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, \\ X^+ < Y^+ \}$$



Why Do We Have Completeness?

The set \mathcal{C} is **closed** under intersection, composition, and converse (it is a *sub-algebra* wrt. these three operations on relations). This can be shown by using a computer program.

Lemma

Each 3-consistent interval CSP over \mathcal{C} is globally consistent.

Theorem (van Beek)

Path consistency solves $\text{CMIN}(\mathcal{C})$ and decides $\text{CSAT}(\mathcal{C})$.

Proof.

Follows from the above lemma and the fact that a strongly n -consistent CSP is minimal. □

Corollary

A path consistent interval CSP consisting of base relations only is satisfiable.

Helly's Theorem

Definition

A set $M \subseteq \mathbf{R}^n$ is *convex* iff for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M .

Theorem (Helly)

Let F be a family of at least $n + 1$ convex sets in \mathbf{R}^n . If all sub-families of F with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

Strong n -Consistency (1)

Proof.

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3$ \checkmark

Induction assumption: Assume strong $k - 1$ -consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k - 1$ variables.

We have to show that we can extend the instantiation to any k th variable.

Strong n -Consistency (2): Instantiating the k th Variable

Proof (Part 2).

The instantiation of the $k - 1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k .

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \dots$$

Such inequalities define convex subsets in \mathbf{R}^2 .

\rightsquigarrow Consider sets of 3 inequalities (= 3 convex sets).

Strong n -Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+ , but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by the same argument as above there exists a common point.

- ↪ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.
- ↪ Strong k -consistency for all $k \leq n$.

Outlook

- ▶ $\text{CMIN}(\mathcal{C})$ can be computed in $O(n^3)$ time (for n being the number of intervals) using the path consistency algorithm.
- ▶ \mathcal{C} is a set of relations occurring “naturally” when observations are uncertain.
- ▶ \mathcal{C} contains 83 relations (incl. the impossible and the universal relations).
- ▶ Are there larger sets such that path consistency computes minimal CSPs? **Probably not**
- ▶ Are there larger sets of relations that permit polynomial satisfiability testing? **Yes**

A Maximal Tractable Sub-Algebra

Allen's Interval Calculus

Reasoning in Allen's Interval Calculus

A Maximal Tractable Sub-Algebra

- The Endpoint Subclass

- The ORD-Horn Subclass

- Maximality

- Solving Arbitrary Allen CSPs

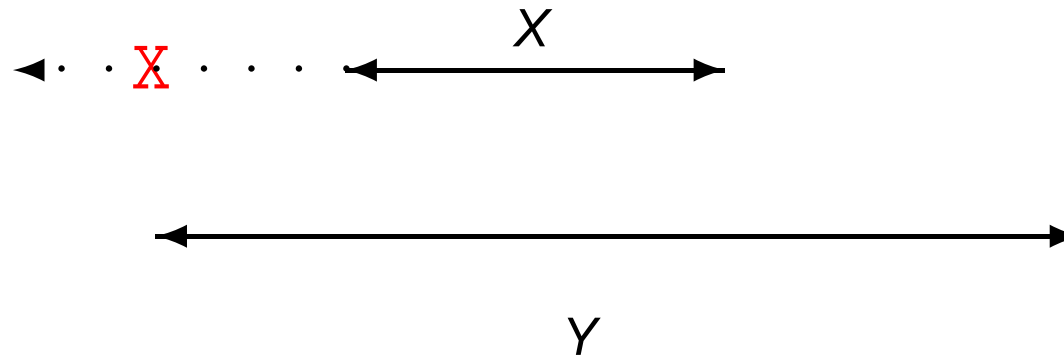
Literature

The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only **unit** clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(X \{d, o\} Y) = \left\{ \begin{array}{l} X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ X^+ < Y^+ \end{array} \right\}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

The path-consistency method decides CSAT(\mathcal{P}).

The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only **Horn clauses**, where only the following **literals** are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{o, s, f^{-1}\}$:

$$\pi(X\{o, s, f^{-1}\}Y) = \left\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \right\}.$$

Partial Orders: The *ORD* Theory

Let *ORD* be the following theory:

$$\begin{array}{llll} \forall x, y, z: & x \leq y \wedge y \leq z & \longrightarrow & x \leq z & \text{(transitivity)} \\ \forall x: & x \leq x & & & \text{(reflexivity)} \\ \forall x, y: & x \leq y \wedge y \leq x & \longrightarrow & x = y & \text{(anti-symmetry)} \\ \forall x, y: & x = y & \longrightarrow & x \leq y & \text{(weakening of =)} \\ \forall x, y: & x = y & \longrightarrow & y \leq x & \text{(weakening of =)}. \end{array}$$

- ▶ *ORD* describes partially ordered sets, \leq being the ordering relation.
- ▶ *ORD* is a [Horn theory](#)
- ▶ What is missing wrt to *dense* and *linear* orders?

Satisfiability over Partial Orders

Proposition

Let Θ be a CSP over \mathcal{H} . Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

\Rightarrow : Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.

\Leftarrow : Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

Complexity of $\text{CSAT}(\mathcal{H})$

Let $\text{ORD}_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$\text{ORD} \cup \pi(\Theta)$ is satisfiable iff $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!



Theorem

$\text{CSAT}(\mathcal{H})$ can be decided in polynomial time.

Proof.

$\text{CSAT}(\mathcal{H})$ instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time.



$$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H} \quad \text{with} \quad |\mathcal{C}| = 83, |\mathcal{P}| = 188, |\mathcal{H}| = 868$$

Path-Consistency and the OH-Class

Lemma

Let Θ be a path-consistent set over \mathcal{H} . Then

$$(X \{ \} Y) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

Proof Idea.

One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt **positive unit resolution**. Since this inference rule is refutation complete for Horn theories, the claim follows. □

Lemma

\mathcal{H} is closed under intersection, composition, and conversion.

Theorem

The path-consistency method decides $CSAT(\mathcal{H})$.

↪ Maximality of \mathcal{H} ?

↪ Do we have to check all 8192 - 868 extensions?

Complexity of Sub-Algebras

Let \hat{S} be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by S)

Theorem

CSAT(\hat{S}) can be polynomially transformed to CSAT(S).

Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in S , introducing perhaps some fresh variables. □

- ↪ Polynomiality of S extends to \hat{S} .
- ↪ NP-hardness of \hat{S} is inherited by all generating sets S .
- ↪ **Note:** $\mathcal{H} = \hat{\mathcal{H}}$.

Minimal Extensions of the \mathcal{H} -Subclass

A computer-aided case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras that strictly contain \mathcal{H} : $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

Theorem

$CSAT(\mathcal{H} \cup \{N_i\})$ is NP-complete.

Question: Are there other maximal tractable subclasses?

“Interesting” Subclasses

Interesting subclasses of \mathcal{A} should contain all basic relations.

A *computer-aided* case analysis reveals: For $S \supseteq \{\{B\} : B \in \mathbf{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. N_1 or N_2 is in \hat{S} .

In case 2, one can show: $\text{CSAT}(S)$ is NP-complete.

\rightsquigarrow \mathcal{H} is the **only** maximal tractable subclass that is **interesting**.

Meanwhile, there is a **complete classification** of all sub-algebras containing at least one basic relation [IJCAI 2001] . . . but the question for sub-algebras not containing a basic relation is open.

Relevance?

Theoretical:

We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*. ⊕

Practical: All known applications either need only \mathcal{P} or they need more than \mathcal{H} ! ⊖

Backtracking methods might profit from the result because the branching factor is lower. ?

↪ How difficult is CSAT(\mathcal{A}) in practice?

↪ What are the relevant branching factors?

Solving General Allen CSPs

- ▶ Backtracking algorithm using **path-consistency** as a *forward-checking method*
 - ▶ Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
 - ▶ Refinements and evaluation of different heuristics
- ~> Which tractable fragment should one use?

Branching Factors

- ▶ If the labels are split into **base relations**, then on average a label is split into

6.5 relations

- ▶ If the labels are split into **pointizable relations** (\mathcal{P}), then on average a label is split into

2.955 relations

- ▶ If the labels are split into **ORD-Horn relations** (\mathcal{H}), then on average a label is split into

2.533 relations

↪ A difference of **0.422**

↪ This makes a difference for “hard” instances.

Summary

- ▶ Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- ▶ The satisfiability problem for CSPs using the relations is NP-complete.
- ▶ For the **continuous endpoint class**, minimal CSPs can be computed using the path-consistency method.
- ▶ For the larger **ORD-Horn class**, CSAT is still decided by the path-consistency method.
- ▶ Can be used in practice for backtracking algorithms.

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