

Qualitative Representation and Reasoning

Introduction

Knowledge Representation and Reasoning

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Quantitative vs. Qualitative

Spatio-temporal configurations can be described **quantitatively** by specifying the coordinates of the relevant objects:

Example: *At time point 10.0 object A is at position (11.0, 1.0, 23.7), at time point 11.0 at position (15.2, 3.5, 23.7). From time point 0.0 to 11.0, object B is at position (15.2, 3.5, 23.7). Object C is at time point 11.0 at position (300.9, 25.6, 200.0) and at time point 35.0 at (11.0, 1.0, 23.7).*

Often, however, a **qualitative** description (using a finite vocabulary) is more adequate:

Example: *Object A hit object B. Afterwards, object C arrived.*

Sometimes we want to reason with such descriptions, e.g.:

Object C was not close to object A when it hit object B.

Representation of Qualitative Knowledge

Intention: Description of configurations using a finite vocabulary and reasoning about these descriptions

- ▶ Specification of a **vocabulary**: usually a finite set of relations (often binary) that are **pairwise disjoint** and **exhaustive**
- ▶ Specification of a **language**: often sets of atomic formulae (constraint networks), perhaps restricted disjunction
- ▶ Specification of a formal **semantics**
- ▶ Analysis of computational properties and design of **reasoning methods** (often constraint propagation)
- ▶ Perhaps, specification of **operational semantics** for verifying whether a relation holds in a given quantitative configuration

Applications in ...

- ▶ Natural language processing
- ▶ Specification of abstract spatio-temporal configurations
- ▶ Query languages for spatio-temporal information systems
- ▶ Layout descriptions of documents (and learning of such layouts)
- ▶ Action planning
- ▶ ...

Qualitative Temporal Relations: *Point Calculus*

We want to talk about **time instants** (points) and binary **relations** over them.

► *Vocabulary:*

- X equals Y : $X = Y$
- X before Y : $X < Y$
- X after Y : $X > Y$

► *Language:*

- Allow for **disjunctions** of basic relations to express **indefinite information**. Use set of relations to express that. For instance, $\{<, =\}$ expresses \leq .
- 2^3 different relations (including the *impossible* and the *universal* relation)
- Use **sets of atomic formulae** with these relations to describe **configurations**. For example:

$$\{x\{=\}y, y\{<, >\}z\}$$

- *Semantics*: Interpret the time point symbols and relation symbols over the **rational** (or real) numbers.

Some Reasoning Problems

$$\left\{ x\{<,=\}y, y\{<,=\}z, v\{<,=\}y, w\{>\}y, z\{<,=\}x \right\}$$

- ▶ **Satisfiability**: Are there values for all time points such that all formulae are satisfied?
- ▶ **Satisfiability** with $v\{=\}w$?
- ▶ Finding a satisfying **instantiation** of all time points
- ▶ **Deduction**: Does $x\{=\}y$ logically follow?
Does $v\{<,=\}w$ follow?
- ▶ Finding a **minimal description**: What are the most constrained relations that describe the same set of instantiations?

From a Logical Point of View . . .

In general, qualitatively described configurations are simple logical theories:

- ▶ Only sets of atomic formulae to describe the configuration
- ▶ Only existentially quantified variables (or constants)
- ▶ A fixed background theory that describes the semantics of the relations (e.g., dense linear orders)
- ▶ We are interested in **satisfiability**, **model finding**, and *deduction*
- ▶ *Constraint Satisfaction Problems*

CSP – Definition

Definition

A *constraint satisfaction problem (CSP)* is given by

- ▶ a set V of n variables $\{v_1, \dots, v_n\}$,
- ▶ for each v_i , a value domain D_i
- ▶ constraints (relations over subsets of the variables)

Tasks:

Find one (or all) *solution(s)*, i. e., tuples

$$(d_1, \dots, d_n) \in D_1 \times \dots \times D_n$$

such that the assignment $v_i \mapsto d_i$ ($1 \leq i \leq n$) satisfies all constraints.

CSP – Example

k -colorability: Can we color the nodes of a graph with k colors in a way such that all nodes connected by an edge have different colors?

- ▶ The node set is the set of variables
- ▶ The domain of each variable is $\{1, \dots, k\}$
- ▶ The constraints are that nodes connected by an edge must have a different value

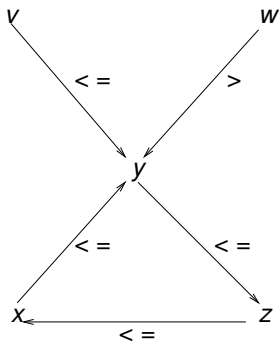
Note: This CSP has a particular restricted form:

- ▶ Only binary constraints
- ▶ The domains are finite

Other examples: Many problems (e.g. cross-word puzzle, n -queens problem, configuration, ...) can be cast as a CSP (and solved this way)

Our Example: Point relations

- ▶ Our point relation CSP is a binary CSP with **infinite domains**.
- ▶ It can be represented as a **constraint graph**:



Computational Complexity

Theorem

It is NP-hard to decide solvability of CSPs, even binary CSPs.

Proof.

Since k -colorability is NP-complete (even for fixed $k \geq 3$), solvability of CSPs in general must be NP-hard. □

Question: Is CSP solvability *in* NP?

Solving CSP

- ▶ **Enumeration** of all assignments and testing
- ↪ ... too costly
- ▶ **Backtracking** search
- ↪ 1001 different strategies, often “dead” search paths are explored extensively
- ▶ **Constraint propagation**: elimination of obviously impossible values followed by backtracking search
- ▶ Many other search methods, e.g., local search, stochastic search, etc.
- ↪ How do we solve CSP with infinite domains?

General Assumptions

- ▶ Only at most **binary** constraints (i.e., we can use **constraint graph**)
- ▶ Uniform domain D for all variables
- ▶ Unary constraints D_i and binary constraints R_{ij} are **sets** of values or sets of pairs of values, resp.
- ▶ We assume that for all nodes i, j :

$$(x, y) \in R_{ij} \Rightarrow (y, x) \in R_{ji}$$

Local Consistency

- ▶ A CSP is *locally consistent* if for particular subsets of the variables, solutions of the restricted CSP can be extended to solutions of a larger set of variables.
- ↪ Methods to transform a CSP into a tighter, but “equivalent” problem.

Definition

A binary CSP $\langle V, D, C \rangle$ is *arc consistent* (or *2-consistent*) if for all nodes $1 \leq i, j \leq n$,

$$x \in D_i \Rightarrow \exists y \in D_j \text{ s. t. } (x, y) \in R_{ij}$$

- ↪ When a CSP is *arc consistent*, each one variable assignment $\{v_i\} \rightarrow D$ that satisfies all (unary) constraints in v_i , i. e., D_i , can be extended to a two variable assignment $\{v_i, v_j\} \rightarrow D$ that satisfies all unary/binary constraints in these variables, i. e., D_i , D_j , and R_{ij} .

Arc Consistency

EnforceArcConsistency(\mathcal{C}):

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$

Output: an equivalent, but arc consistent CSP \mathcal{C}'

repeat

for each arc (v_i, v_j) with $R_{ij} \in C$

$D_i := D_i \cap \{x \in D : \text{ex. } y \in D_j \text{ s.t. } (x, y) \in R_{ij}\}$

endfor

until no domain is changed

- ▶ Terminates in time $O(n^3 \cdot k^3)$ if we have finite domains (where k is the number of values)
- ↪ There exist different (more efficient) algorithms for enforcing arc consistency.

Arc Consistency

Lemma

- ▶ *Enforcing arc consistency yields an arc consistent CSP.*
 - ▶ *Enforcing arc consistency is solution invariant, i. e. it does not change the set of solutions.*
- ↪ Arc consistent CSPs need not be consistent, and vice versa.

Arc Consistency – Example

$$D_1 = \{1, 2, 3\}$$

$$D_2 = \{2, 3\}$$

$$D_3 = \{2\}$$

$$R_{ij} = "\neq" \text{ for } i \neq j$$

1. $D_1 := D_1 \cap \{x : y \in D_3 \wedge (x, y) \in R_{13}\} = \{1, 3\}$
2. $D_2 := D_2 \cap \{x : y \in D_3 \wedge (x, y) \in R_{23}\} = \{3\}$
3. $D_1 := D_1 \cap \{x : y \in D_2 \wedge (x, y) \in R_{12}\} = \{1\}$
4. CSP is now **arc consistent**

- ▶ Since all unary constraints are singletons, this defines a **solution** of the CSP.
- ▶ Since enforcing arc consistency does not change the set of solutions, this is a unique solution of the original CSP.

Local Consistency (2): Path Consistency

Definition

A binary CSP $\langle V, D, C \rangle$ is said to be *path consistent* (or *3-consistent*) if for all nodes $1 \leq i, j, k \leq n$,

$$x \in D_i, y \in D_j, (x, y) \in R_{ij} \Rightarrow \\ \exists z \in D_k \text{ s. t. } (x, z) \in R_{ik} \text{ and } (y, z) \in R_{jk}$$

- ↪ When a CSP is *path consistent*, each two variable assignment $\{v_i, v_j\} \rightarrow D$ satisfying all constraints in v_i and v_j can be extended to any three variable assignment $\{v_i, v_j, v_k\} \rightarrow D$ such that all constraints in these variables are satisfied.

Path Consistency

EnforcePathConsistency(\mathcal{C}):

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$ of size n

Output: a equivalent, but path consistent CSP \mathcal{C}'

repeat

for all $1 \leq i, j, k \leq n$

$R_{ij} := R_{ij} \cap$

$\{(x, y) : \text{ex. } z \in D_k \text{ s. t. } (x, z) \in R_{ik} \text{ and } (y, z) \in R_{jk}\}$

endfor

until no binary constraint is changed

- ↪ Terminates in time $O(n^5 \cdot k^5)$ if we have finite domains (where k is the number of values)
- ↪ Enforcing path consistency is solution invariant.

Local Consistency (3): k -Consistency and Strong k -Consistency

Definition

- ▶ A binary CSP $\langle V, D, C \rangle$ is *k -consistent* if, given variables x_1, \dots, x_k and an assignment $a : \{x_1, \dots, x_{k-1}\} \rightarrow D$ that satisfies all constraint in these variables, a can be extended to an assignment $a' : \{x_1, \dots, x_k\} \rightarrow D$ that satisfies all constraints in these k variables.
- ▶ A binary CSP $\langle V, D, C \rangle$ is *strongly k -consistent* if it is k' -consistent for each $k' \leq k$.
- ▶ A binary CSP $\langle V, D, C \rangle$ is *globally consistent* if it is strongly n -consistent where n is the size of V .

Local Consistency (3)

- ▶ k -consistency: The computation costs grow exponentially with k .
- ▶ If a CSP is globally consistent, then
 - ▶ a solution can be constructed in polynomial time,
 - ▶ its constraints are **minimal**,
 - ▶ and it has a solution iff there is no empty constraint.
- ▶ k -consistent $\not\Rightarrow k - 1$ -consistent

Qualitative Reasoning with CSP

If we want to use CSPs for qualitative reasoning, we have

- ▶ **infinite** domains
- ▶ mostly only **finitely many** relations (basic relations and their unions)
- ▶ **arc consistent** CSPs (usually)

Questions:

- ▶ How do we achieve **k -consistency** (for some fixed k)?
- ▶ Is k -consistency (for some fixed k) enough to guarantee **global consistency**?
- ▶ Is a CSP with only **base relations** always satisfiable?

Operations on Binary Relations

Composition:

$$R_1 \circ R_2 = \{(x, y) \in D^2 : \exists z \in D \text{ s.t. } (x, z) \in R_1 \text{ and } (z, y) \in R_2\}$$

Converse:

$$R^{-1} = \{(x, y) \in D^2 : (y, x) \in R\}$$

Intersection:

$$R_1 \cap R_2 = \{(x, y) \in D^2 : (x, y) \in R_1 \text{ and } (x, y) \in R_2\}$$

Union:

$$R_1 \cup R_2 = \{(x, y) \in D^2 : (x, y) \in R_1 \text{ or } (x, y) \in R_2\}$$

Complement:

$$\overline{R} = \{(x, y) \in D^2 : (x, y) \notin R\}$$

Conditions on Vocabulary for Qualitative Reasoning

- ▶ Let \mathbf{B} be a finite set of (binary) **base relations**.
- ↪ The relations in \mathbf{B} should be **JEPD**, i. e., jointly exhaustive and pairwise disjoint.
- ▶ \mathbf{B} should be **closed under converse**.
- ▶ Let \mathbf{A} be the set of relations that can be built by taking the unions of relations from \mathbf{B} (↪ $2^{|\mathbf{B}|}$ different relations).
- ↪ \mathbf{A} is closed under converse, complement, intersection and union.
- ▶ \mathbf{A} should be **closed under composition of base relations**, i. e., for all $B, B' \in \mathbf{B}$, $B \circ B' \in \mathbf{A}$.
- ↪ \mathbf{A} is closed under composition of arbitrary relations.
- ↪ This condition does not hold necessarily.

Example: $\mathbf{B} = \{<, =, >\}$ interpreted over the integers is not closed under composition (and has no finite closure):

$$< \circ < = < \setminus \{(i, j) : i = j - 1\} \subsetneq <$$

Computing Operations on Relations

Let \mathbf{A} be a relation system over the set of **base relations** \mathbf{B} that satisfies the conditions spelled out above.

↪ We may write relations as *sets* of base relations:

$$B_1 \cup \dots \cup B_n \sim \{B_1, \dots, B_n\}$$

Then the operations on the relations can be *computed* as follows:

Composition:

$$\{B_1, \dots, B_n\} \circ \{B'_1, \dots, B'_m\} = \bigcup_{i=1}^n \bigcup_{j=1}^m B_i \circ B'_j$$

Converse:

$$\{B_1, \dots, B_n\}^{-1} = \{B_1^{-1}, \dots, B_n^{-1}\}$$

Complement:

$$\overline{\{B_1, \dots, B_n\}} = \{B \in \mathbf{B} : B \neq B_i, \text{ for each } 1 \leq i \leq n\}$$

Intersection and **union** are defined set-theoretically.

Reasoning Problems

Given a qualitative CSP:

CSP-Satisfiability (CSAT):

- ▶ Is the CSP satisfiable/solvable?

CSP-Entailment (CENT):

- ▶ Given in addition xRy : Is xRy satisfied in each solution of the CSP?

Computation of an equivalent minimal CSPs (CMIN):

- ▶ Compute for each pair x, y the strongest constrained (minimal) relation entailed by the CSP.

↪ These problems are equivalent under **Turing reductions**

Turing Reductions

Ordinary *polynomial reductions*:

$$A \leq B \text{ iff ex. } f \text{ computable in poly. time s.t.} \\ x \in A \Leftrightarrow f(x) \in B.$$

Definition

A is *Turing-reducible* to B iff A can be decided in poly. time by using a procedure for deciding B :

$$A \leq_T B \text{ iff } A \text{ can be decided in poly.} \\ \text{time by a procedure for } B.$$

- ▶ Note: \leq implies \leq_T , but not the other way around.

Reductions between CSP Problems

Theorem

CSAT, CENT and CMIN are equivalent under polynomial Turing reductions.

Proof.

$CSAT \leq_T CENT$ and $CENT \leq_T CMIN$ are obvious.

$CENT \leq_T CSAT$: We solve $CENT$ ($CSP \models xRy?$) by testing satisfiability of the CSP extended by $x\{B\}y$ where B ranges over all base relations. Let B_1, \dots, B_k be the relations for which we get a positive answer. Then $x\{B_1, \dots, B_k\}y$ is entailed by the CSP.

$CMIN \leq_T CENT$: We use entailment for computing the minimal constraint for each pair. Starting with the universal relation, we remove one base relation until we have a minimal relation that is still entailed. □

Path Consistency for Qualitative CSPs

Given a qualitative CSP with $R_{ij} = R_{ji}^{-1}$. Then **path consistency** can be enforced by doing the following:

$$R_{ij} := R_{ij} \cap (R_{ik} \circ R_{kj}).$$

Path consistency guarantees ...

- ▶ sometimes **minimality**
 - ▶ sometimes **satisfiability**
 - ▶ however sometimes the CSP is **not satisfiable**, even if the CSP contains only **base relations**
- ↪ All this depends on the vocabulary.

Example: Point Relations

Composition table:

	<	=	>
<	<	<	<, =, >
=	<	=	>
>	<, =, >	>	>

Figure: Composition table for the point algebra. For example:
 $\{<\} \circ \{=\} = \{<\}$

- ▶ $\{<, =\} \circ \{<\} = \{<\}$
- ▶ $\{<, >\} \circ \{<\} = \{<, =, >\}$
- ▶ $\{<, =\}^{-1} = \{>, =\}$
- ▶ $\{<, =\} \cap \{>, =\} = \{=\}$

Some Properties of the Point Relations

Theorem

A path consistent CSP over the point relations is consistent.

Corollary

CSAT, CENT and CMIN are polynomial problems for the point relations.

Theorem

A path consistent CSP over all point relations without $\{<, >\}$ is minimal.

Proofs later ...

A Pathological Relation System

Let e, d, i be (self-converse) base relations between points on a circle:

- ▶ e : Rotation by 72 degrees (left or right)
- ▶ d : Rotation by 144 degrees (left or right)
- ▶ i : Identity

Composition table:

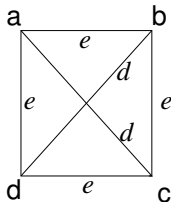
$$e \circ e = \{i, d\}$$

$$d \circ d = \{i, e\}$$

$$e \circ d = \{e, d\}$$

$$d \circ e = \{e, d\}$$

The following CSP is path consistent and contains only base relations, but it is not satisfiable:



Outlook

- ▶ **Qualitative representation and reasoning** usually starts with a finite vocabulary (a finite set of relations).
- ▶ Qualitative descriptions are usually simply logical theories consisting of sets of atomic formulae (and some background theory).
- ▶ **Reasoning problems** are (as usual) satisfiability, model finding, and deduction.
- ▶ Can be addressed with **CSP methods** (but note: **infinite** domains).
- ▶ **Path consistency** is the basic reasoning step . . . sometimes this is enough.
- ▶ Usually, path-consistent atomic CSPs are satisfiable. However, there exist some pathological relation systems.
- ▶ Can be taken further \rightsquigarrow **relation algebra**

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