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# Belief Revision

- Changing our beliefs as the world or our information about it changes is belief revision.
- We start with some belief state $K$ (a deductively closed set of formulae). When new information arrives, we change the belief state in order to *accommodate the new information*.
- If the new information contradicts the old beliefs, the new belief state is not (monotonically) a superset of the old belief state.
- Contrary to nonmonotonic reasoning, here we deal with temporal nonmonotonicity, i.e., the nonmonotonic evolution of a knowledge base or belief state over time.





# Two Scenarios: Update and Revision

- We have a theory about the world, and the new information is meant to *correct* our theory.

$\rightsquigarrow$ belief revision: change your belief state minimally in order to accommodate the new information.

- We have a correct theory about the current state of the world, and the new information is meant to record a *change* in the world.

$\rightsquigarrow$ belief update: incorporate the change by assuming that the world has changed minimally.





# Update and Revision are Different

Assume the new information is consistent with our old beliefs.

- In case of revision, we would like to add the new information monotonically to our old beliefs.
- For belief update this is not necessarily the case.
  - We know that the *door is open or the window is open*.
  - We learn that the world has changed and the *door is now closed*.
  - In this case, we do not want to add this information monotonically to our theory, since we would be forced to conclude that *the window is open*.

# Belief Change Operations

- General assumption: A belief state is modeled by a deductively closed theory $K$, i.e., $K = Th(K)$.
- $\mathcal{L}$: Logical language (propositional logic)
- $Th_{\mathcal{L}}$: Set of deductively closed theories (or belief sets) over $\mathcal{L}$
- Belief Change Operations:

Monotonic addition: $+ : \text{Th}_{\mathcal{L}} \times \mathcal{L} \rightarrow \text{Th}_{\mathcal{L}}$

$K + \psi = \text{Th}(K \cup \{\psi\})$

Revision: $\dot{+} : \text{Th}_{\mathcal{L}} \times \mathcal{L} \rightarrow \text{Th}_{\mathcal{L}}$

- *Reasonable* revision operations?
- *AGM Revision Postulates* (Alchourron, Gärdenfors, Makinson)





# AGM Postulates: Constraining the space of Revision Operations

($\dot{+}$1) $K \dot{+} \varphi \in \text{Th}_{\mathcal{L}}$

($\dot{+}$2) $\varphi \in K \dot{+} \varphi$

($\dot{+}$3) $K \dot{+} \varphi \subseteq K + \varphi$

($\dot{+}$4) If $\neg\varphi \notin K$ then $K + \varphi \subseteq K \dot{+} \varphi$

($\dot{+}$5) $K \dot{+} \varphi = \text{Th}(\bot)$ only if $\models \neg\varphi$

($\dot{+}$6) If $\models \varphi \leftrightarrow \psi$ then $K \dot{+} \varphi = K \dot{+} \psi$

($\dot{+}$7) $K \dot{+} (\varphi \wedge \psi) \subseteq (K \dot{+} \varphi) + \psi$

($\dot{+}$8) If $\neg\psi \notin K \dot{+} \varphi$
then $(K \dot{+} \varphi) + \psi \subseteq K \dot{+} (\varphi \wedge \psi)$





# Canonical Revision Operations?

- The postulates constrain the space of revision operations, but do not choose one uniquely.
- Revision operations are closed under intersection, so should we choose the minimum?
- NO! This is *full meet revision*, which is useless since $K \dot{+} \phi = \text{Th}(\phi)$ for all $\phi$ that are inconsistent with $K$:

  *For every $\psi$ such that $\phi \not\models \psi$ there are revision operations such that $\psi \in K \dot{+} \phi$ and operations such that $\neg\psi \in K \dot{+} \phi$. Hence $\psi$ is not in the intersection of all revisions.*

  $\Longrightarrow$ *AGM postulates alone are too weak!*





# Revision with Preference Information

- AGM postulates allow giving up *any beliefs* that do not follow from the new information.
- This has to be prevented: Some beliefs are *preferred to* others.
- We need revision operations that use preference information on beliefs.

# Examples

Partial Meet Revision (AGM): Preference information is given by a *selection function* $\gamma$ over the set of maximal consistent subtheories $(K \downarrow \neg\varphi)$ of $K$ that do not include $\neg\varphi$:

$$K \dot{+} \varphi \stackrel{\text{def}}{=} \left(\bigcap \gamma(K \downarrow \neg\varphi)\right) + \varphi.$$

Cut Revision (GM): Preference information is given by complete preorder $\preceq$ over all $\psi \in K$:

$$K \dot{+} \varphi \stackrel{\text{def}}{=} \{\psi \in K \mid \neg\varphi \prec \psi\} + \varphi$$

Provided that $\preceq$ satisfies certain axioms, cut revisions satisfy AGM postulates.





# Revision – Viewed Computationally

- We don't want to deal with deductively closed (infinite) theories.
- Consider belief bases (finite sets of formulae) as *representing* belief sets.
- A theory $K$ over the propositional logic $\mathcal{L}$ with $n$ propositional atoms can have as many as
  - $2^{2^n}$ different propositions and
  - $2^n$ different models.
- Consider ways of specifying preference information in a *concise* way, i.e., polynomial in the size of the belief base.





# Base Revision

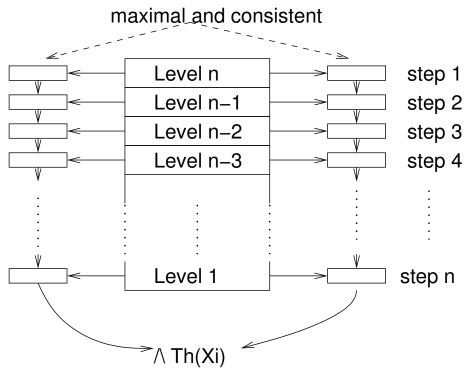
- We start from a finite belief base $A$ and preference information over the elements of $A$.
- We want to generate a revision operation (restricted to Th($A$)).
- Assume a partitioning of $A$ into $n$ *priority classes* $A_1, \ldots A_n$ such that the elements of $A_i$ are more important/preferred than those of $A_j$ for $i > j$.
- Define a revision operation that keeps as many of the more important formulae as possible.





# Prioritized Meet Base Revision (PMBR)

Prioritized Meet Base Revision is defined by maximizing the most important formulae.



# Prioritized Meet Base Revision (PMBR) – Formally

**Definition ($\Downarrow$)**

Let $A$ be a base with partitioning $A_1, \ldots, A_n$.
$(A \Downarrow \varphi)$ consists of sets $S = S_1 \cup \cdots \cup S_n$ fulfilling the following for all $i \in \{n, \ldots, 1\}$.

$$
\begin{aligned}
S_{n+1} &= \emptyset \\
S_i &\text{ is a maximal subset of } A_i \cup S_{i+1} \text{ such that } \\
&S_{i+1} \subseteq S_i \text{ and } S_i \not\models \varphi
\end{aligned}
$$





# Prioritized Meet Base Revision – Example of $\Downarrow$

Compute one element of $A \Downarrow \neg F$.

$$
\begin{aligned}
A_4 &= \{P \rightarrow B, B \rightarrow F\} & S_4 &= \{P \rightarrow B, B \rightarrow F\} \\
A_3 &= \{P \rightarrow \neg F\} & S_3 &= \{P \rightarrow \neg F\} \cup S_4 \\
A_2 &= \{B\} & S_2 &= \{B\} \cup S_3 \\
A_1 &= \{P\} & S_1 &= \{\} \cup S_2
\end{aligned}
$$

Actually, $S_1 \cup S_2 \cup S_3 \cup S_4$ is the only element of

$$A \Downarrow \neg F = \{\{P \rightarrow B, B \rightarrow F, P \rightarrow \neg F, B\}\}.$$





# Prioritized Meet Base Revision (PMBR) – Formally

**Definition**

$$A \oplus \varphi \overset{\text{def}}{=} \Big( \bigcap_{B \in (A \Downarrow \neg\varphi)} \mathrm{Th}(B) \Big) + \varphi.$$

**Definition**

Revision operation $\dot{+}$ on $\mathrm{Th}(A)$ (that depends on $A$ and the priority information) is

$$\mathrm{Th}(A) \dot{+} \varphi \overset{\text{def}}{=} A \oplus \varphi.$$





# Properties of PMBRs

- Generates *partial meet revision*, but does not satisfy $(\dot{+}8)$ in general.
- Deciding whether $A \oplus \varphi \models \psi$ is $\Pi_2^p$-complete, even for one priority class. (Proof is similar to $\Pi_2^p$-completeness of PDS for default logic.)
- A revised base can be represented by

$$A \oplus \varphi = \mathrm{Th}\Big( (\bigvee (A \Downarrow \neg\varphi)) \wedge \varphi \Big).$$

- A revised base can become exponentially large:

$$
\begin{aligned}
A &= \{p_1, \ldots, p_m, q_1, \ldots, q_m\} \\
\varphi &= \bigwedge_{i=1}^{m} (p_i \leftrightarrow \neg q_i)
\end{aligned}
$$

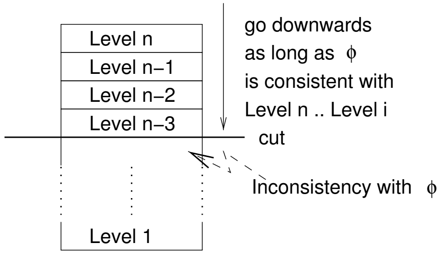
$(A \Downarrow \varphi)$ has size exponential in $|A|$.

# Cut Base Revision (CBR)

## Definition

Let $\widehat{A_j} \stackrel{\text{def}}{=} \bigcup_{i=j}^{n} A_i$, then cut base revision $\otimes$ is defined as:

$$A \otimes \varphi \stackrel{\text{def}}{=} \text{Th}(\bigcup_{j=1}^{n} \{\psi \in A_j | \widehat{A_j} \not\models \neg\varphi\}) + \varphi.$$




- *Easy* to compute: in $\mathrm{P}^{\mathrm{NP}[O(\log n)]}$.
- With Horn logic only $O(n \log n)$.

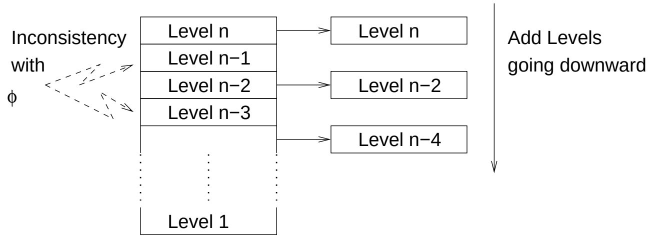




# Less conservative Linear Base Revision (LBR)

## Definition (Idea informally)

Throw away an entire priority class only if it would lead to a *contradiction* which cannot be blamed on a lower classes $\rightsquigarrow$ *linear base-revision* $\odot$.




- Generates revision operations satisfying the AGM postulates.
- Complexity: $\Delta_2^p$-complete; $O(n^2)$ for Horn logic.
- *LBR* $\approx$ *CBR*, but a CBR realizing an LBR requires exponentially more priority classes.





# Revision vs. Nonmonotonic Reasoning

Belief Revision and Nonmonotonic Reasoning seem to be of different nature, but there is a tight connection:

- Given $K$ and a revision operation $\dot{+}$

$\rightsquigarrow$ a nonmonotonic consequence relation can be defined as follows: $\phi \mid\!\sim \psi$ iff $\psi \in K \dot{+} \phi$.

In this case,

- the rationality postulates correspond to principles of NMR (such as cautious monotony etc.);
- in the case of prerequisite-free, normal defaults $D$, the cautions conclusions from $(W, D)$ are simply $D \oplus W$ with one priority level.





# NMR Principles and Rationality Postulates

($\dot{+}$2) $\varphi \in K \dot{+} \varphi$;
- *Reflexivity*

($\dot{+}$3) $K \dot{+} \varphi \subseteq K + \varphi$;
- *Supraclassicality*

($\dot{+}$6) If $\models \varphi \leftrightarrow \psi$ then $K \dot{+} \varphi = K \dot{+} \psi$;
- *Left Logical Equivalence*

($\dot{+}$8) If $\neg\psi \notin K \dot{+} \varphi$, then $(K \dot{+} \varphi) + \psi \subseteq K \dot{+} (\varphi \wedge \psi)$.
- *Rational Monotonicity*

# Outlook & Summary

- While NMR and belief revision seem to be two sides of the same coin, there are notable *pragmatic differences*:
  - Belief revision seems to require that we can easily represent the changed belief base, while for NMR it makes sense to use dense representations.
  - A similar argument could be made for the computational complexity.
- NMR and Belief Revision can be thought of as *qualitative ways* of dealing with uncertainty in a purely logical setting.
- There exists a strong *correspondence* between NMR and BR
- Both are computationally expensive and representationally problematic.





# Literature

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