

# Principles of Knowledge Representation and Reasoning

## Description Logics – Algorithms

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# Reasoning Problems & Algorithms

- *Satisfiability* or *subsumption* of concept descriptions
- *Satisfiability* or *instance relation* in ABoxes
- **Structural subsumption algorithms**
  - *Normalization* of concept descriptions and *structural comparison*
  - very fast, but can only be used for small DLs
- **Tableau algorithms**
  - Similar to modal tableau methods
  - Meanwhile the method of choice

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# Structural Subsumption Algorithms

- **Small Logic**  $\mathcal{FL}^-$

- $C \sqcap D$
- $\forall r.C$
- $\exists r$  (simple existential quantification)

- **Idea**

- 1 In the conjunction, collect all *universally quantified expressions* (also called *value restrictions*) with the same role and build *complex value restriction*:

$$\forall r.C \sqcap \forall r.D \rightarrow \forall r.(C \sqcap D).$$

- 2 Compare all conjuncts with each other. For each conjunct in the subsuming concept there should be a *corresponding one* in the subsumed one.

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$$D = \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child}.\text{Human} \sqcap \\ \forall \text{has-child}.\exists \text{has-child}$$

$$C = \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \\ \forall \text{has-child}.\text{(Human} \sqcap \text{Female} \sqcap \exists \text{has-child)}$$

Check:  $C \sqsubseteq D$

- 1 Collect value restrictions in  $D$ :

...  $\forall \text{has-child}.\text{(Human} \sqcap \exists \text{has-child)}$

- 2 Compare:

- 1 For  $\text{Human}$  in  $D$ , we have  $\text{Human}$  in  $C$
- 2 For  $\exists \text{has-child}$  in  $D$ , we have ...
- 3 For  $\forall \text{has-child}.\text{(...)}$  in  $D$ , we have ...
  - 1 For  $\text{Human}$  ...
  - 2 For  $\exists \text{has-child}$  ...

$\rightsquigarrow C$  is subsumed by  $D$ !

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$$\begin{aligned}D &= \text{Human} \sqcap \exists \text{has-child} \sqcap \forall \text{has-child}.\text{Human} \sqcap \\ &\quad \forall \text{has-child}.\exists \text{has-child} \\ C &= \text{Human} \sqcap \text{Female} \sqcap \exists \text{has-child} \sqcap \\ &\quad \forall \text{has-child}.\text{(Human} \sqcap \text{Female} \sqcap \exists \text{has-child)}\end{aligned}$$

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# Subsumption Algorithm

## SUB( $C, D$ ) algorithm:

- 1 Reorder terms (*commutativity*, *associativity* and *value restriction law*):

$$C = \bigwedge A_i \cap \bigwedge \exists r_j \cap \bigwedge \forall r_k : C_k$$

$$D = \bigwedge B_l \cap \bigwedge \exists s_m \cap \bigwedge \forall s_n : D_n$$

- 2 For each  $B_l$  in  $D$ , is there an  $A_i$  in  $C$  with  $A_i = B_l$ ?
  - 3 For each  $\exists s_m$  in  $D$ , is there an  $\exists r_j$  in  $C$  with  $s_m = r_j$ ?
  - 4 For each  $\forall s_n : D_n$  in  $D$ , is there a  $\forall r_k : C_k$  in  $C$  such that  $C_k \sqsubseteq D_n$  and  $s_n = r_k$ ?
- $\rightsquigarrow C \sqsubseteq D$  iff all questions are answered positively

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- 2 For each  $B_l$  in  $D$ , is there an  $A_i$  in  $C$  with  $A_i = B_l$ ?
- 3 For each  $\exists s_m$  in  $D$ , is there an  $\exists r_j$  in  $C$  with  $s_m = r_j$ ?
- 4 For each  $\forall s_n : D_n$  in  $D$ , is there a  $\forall r_k : C_k$  in  $C$  such that  $C_k \sqsubseteq D_n$  and  $s_n = r_k$ ?

$\rightsquigarrow C \sqsubseteq D$  iff all questions are answered positively

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# Subsumption Algorithm

## SUB( $C, D$ ) algorithm:

- 1 Reorder terms (*commutativity*, *associativity* and *value restriction law*):

$$C = \bigwedge A_i \cap \bigwedge \exists r_j \cap \bigwedge \forall r_k : C_k$$

$$D = \bigwedge B_l \cap \bigwedge \exists s_m \cap \bigwedge \forall s_n : D_n$$

- 2 For each  $B_l$  in  $D$ , is there an  $A_i$  in  $C$  with  $A_i = B_l$ ?
  - 3 For each  $\exists s_m$  in  $D$ , is there an  $\exists r_j$  in  $C$  with  $s_m = r_j$ ?
  - 4 For each  $\forall s_n : D_n$  in  $D$ , is there a  $\forall r_k : C_k$  in  $C$  such that  $C_k \sqsubseteq D_n$  and  $s_n = r_k$ ?
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$$SUB(C, D) \Rightarrow C \sqsubseteq D$$

## Proof sketch.

Reordering of terms (1):

a) Commutativity and associativity are trivial

b) Value restriction law. We show:  $(\forall r.(C \sqcap D))^{\mathcal{I}} = (\forall r.C \sqcap \forall r.D)^{\mathcal{I}}$

Assumption:  $d \in (\forall r.(C \sqcap D))^{\mathcal{I}}$

Case 1:  $\nexists e : (d, e) \in r^{\mathcal{I}} \quad \checkmark$

Case 2:  $\exists e : (d, e) \in r^{\mathcal{I}} \Rightarrow e \in (C \sqcap D)^{\mathcal{I}} \Rightarrow e \in C^{\mathcal{I}}, e \in D^{\mathcal{I}}$

Since  $e$  is arbitrary:  $d \in (\forall r.C)^{\mathcal{I}}, d \in (\forall r.D)^{\mathcal{I}}$  then

$d$  must also be conjunction, i.e.,

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Other direction is similar

(2+3+4): Induction on the nesting depth of  $\forall$ -expressions □

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# Completeness

## Theorem (Completeness)

$$C \sqsubseteq D \Rightarrow \text{SUB}(C, D)$$

Proof idea.

One shows the contrapositive:

$$\neg \text{SUB}(C, D) \Rightarrow C \not\sqsubseteq D$$

**Idea:** If one of the rules leads to a negative answer, we use this to construct an interpretation with a special element  $d$  such that

$$d \in C^{\mathcal{I}}, \text{ but } d \notin D^{\mathcal{I}}$$



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# Generalizing the Algorithm

Extensions of  $\mathcal{FL}^-$  by

- $\neg A$  (*atomic negation*),
- $(\leq nr)$ ,  $(\geq nr)$  (*cardinality restrictions*),
- $r \circ s$  (*role composition*)

does not lead to any problems.

**However:** If we use full existential restrictions, then it is very unlikely that we can come up with a *simple* structural subsumption algorithm – having the same flavor as the one above.

*More precisely:* There is (most probably) no algorithm that uses polynomially many reorderings and simplifications and allows for a simple structural comparison

**Reason:** Subsumption for  $\mathcal{FL}^- + \exists r.C$  is NP-hard (Nutt).

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# ABox Reasoning

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**Idea:** *abstraction* + *classification*

- *Complete* AB<sub>ox</sub> by propagating value restrictions to role fillers
- Compute for each object its *most specialized concepts*
- These can then be handled using the ordinary subsumption algorithm

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**Idea:** *abstraction* + *classification*

- **Complete** ABox by propagating value restrictions to role fillers
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# ABox Reasoning

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Literature

**Idea:** *abstraction* + *classification*

- *Complete* AB<sub>ox</sub> by propagating value restrictions to role fillers
- Compute for each object its *most specialized concepts*
- These can then be handled using the ordinary subsumption algorithm

- **Logic**  $\mathcal{ALC}$ 
  - $C \sqcap D$
  - $C \sqcup D$
  - $\neg C$
  - $\forall r.C$
  - $\exists r.C$
- **Idea**: Decide (un-)satisfiability of a concept description  $C$  by trying to *systematically construct* a model for  $C$ . If that is successful,  $C$  is satisfiable. Otherwise  $C$  is unsatisfiable.

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# Example: Subsumption in a TBox

## TBox

Hermaphrodite  $\doteq$  Male  $\sqcap$  Female

Parents-of-sons-and-daughters  $\doteq$

$\exists \text{has-child.Male} \sqcap \exists \text{has-child.Female}$

Parents-of-hermaphrodite  $\doteq \exists \text{has-child.Hermaphrodite}$

## Query

Parents-of-sons-and-daughters  $\sqsubseteq_{\mathcal{T}}$

Parents-of-hermaphrodites

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# Reductions

## 1 *Unfolding*

$\exists \text{has-child.Male} \sqcap \exists \text{has-child.Female} \sqsubseteq$   
 $\exists \text{has-child.}(\text{Male} \sqcap \text{Female})$

## 2 *Reduction to unsatisfiability*

Is  
 $\exists \text{has-child.Male} \sqcap \exists \text{has-child.Female} \sqcap$   
 $\neg(\exists \text{has-child.}(\text{Male} \sqcap \text{Female}))$   
unsatisfiable?

## 3 *Negation normal form* (move negations inside):

$\exists \text{has-child.Male} \sqcap \exists \text{has-child.Female} \sqcap$   
 $\forall \text{has-child.}(\neg \text{Male} \sqcup \neg \text{Female})$

## 4 *Try to construct a model*

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# Model Construction (1)

- ① **Assumption:** There exists an object  $x$  in the interpretation of our concept:

$$x \in (\exists \dots)^{\mathcal{I}}$$

- ② This implies that  $x$  is in the interpretation of all conjuncts:

$$x \in (\exists \text{has-child.Male})^{\mathcal{I}}$$

$$x \in (\exists \text{has-child.Female})^{\mathcal{I}}$$

$$x \in (\forall \text{has-child.}(\neg \text{Male} \sqcup \neg \text{Female}))^{\mathcal{I}}$$

- ③ This implies that there should be objects  $y$  and  $z$  such that  $(x, y) \in \text{has-child}^{\mathcal{I}}$ ,  $(x, z) \in \text{has-child}^{\mathcal{I}}$ ,  $y \in \text{Male}^{\mathcal{I}}$  and  $z \in \text{Female}^{\mathcal{I}}$  and ...

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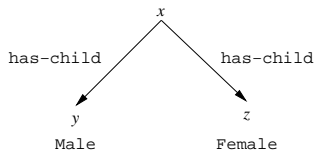
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# Model Construction (2)

$x : \exists \text{has-child.Male}$

$x : \exists \text{has-child.Female}$



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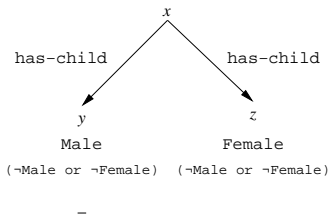
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# Model Construction (3)

$x : \exists \text{has-child.Male}$

$x : \exists \text{has-child.Female}$

$x : \forall \text{has-child.}(\neg \text{Male} \sqcup \neg \text{Female})$



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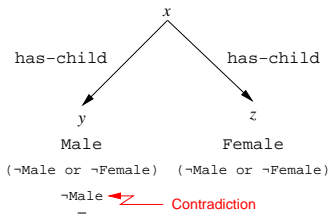
# Model Construction (4)

$x : \exists \text{has-child.Male}$

$x : \exists \text{has-child.Female}$

$x : \forall \text{has-child.}(\neg \text{Male} \sqcup \neg \text{Female})$

$y : \neg \text{Male}$



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# Model Construction (5)

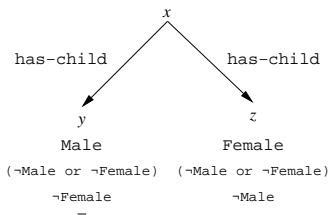
$x : \exists \text{has-child.Male}$

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$x : \forall \text{has-child.}(\neg \text{Male} \sqcup \neg \text{Female})$

$y : \neg \text{Female}$

$z : \neg \text{Male}$



⇒ Model **constructed!**

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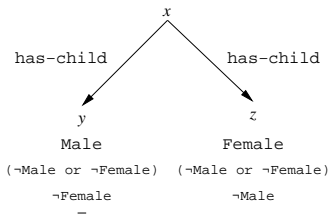
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$\rightsquigarrow$  Model **constructed!**

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# Tableau Method (1): NNF

$C \equiv D$  iff  $C \sqsubseteq D$  and  $D \sqsubseteq C$ .

Now we have the following equivalences:

$$\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$$

$$\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$$

$$\neg\neg C \equiv C$$

$$\neg(\forall r.C) \equiv \exists r.\neg C$$

$$\neg(\exists r.C) \equiv \forall r.\neg C$$

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: **negation normal form (NNF)**

## Theorem (NNF)

*The negation normal form of an  $\mathcal{ALC}$  concept can be computed in polynomial time.*

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# Tableau Method (1): NNF

$C \equiv D$  iff  $C \sqsubseteq D$  and  $D \sqsubseteq C$ .

Now we have the following equivalences:

$$\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$$

$$\neg(C \sqcup D) \equiv \neg C \sqcap \neg D$$

$$\neg\neg C \equiv C$$

$$\neg(\forall r.C) \equiv \exists r.\neg C$$

$$\neg(\exists r.C) \equiv \forall r.\neg C$$

These equivalences can be used to move all negations signs to the inside, resulting in concept description where only concept names are negated: **negation normal form (NNF)**

## Theorem (NNF)

*The negation normal form of an  $\mathcal{ALC}$  concept can be computed in polynomial time.*

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# Tableau Method (2): Constraint Systems

A **constraint** is a syntactical object of the form:  $x: C$  or  $xry$ , where  $C$  is a concept description in NNF,  $r$  is a role name and  $x$  and  $y$  are *variable names*.

Let  $\mathcal{I}$  be an interpretation. An  $\mathcal{I}$ -assignment  $\alpha$  is a function that maps each variable symbol to an object of the universe  $\mathcal{D}$ .

A *constraint*  $x: C$  ( $xry$ ) is **satisfied** by an  $\mathcal{I}$ -assignment  $\alpha$ , if  $\alpha(x) \in C^{\mathcal{I}}$  ( $(\alpha(x), \alpha(y)) \in r^{\mathcal{I}}$ ).

A **constraint system**  $S$  is a finite, non-empty set of constraints. An  $\mathcal{I}$ -assignment  $\alpha$  satisfies  $S$  if  $\alpha$  satisfies each constraint in  $S$ .  $S$  is **satisfiable** if there exists  $\mathcal{I}$  and  $\alpha$  such that  $\alpha$  satisfies  $S$ .

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# Tableau Method (3): Transforming Constraint Systems

## Transformation rules:

- 1  $S \rightarrow_{\sqcap} \{x: C_1, x: C_2\} \cup S$   
if  $(x: C_1 \sqcap C_2) \in S$  and either  $(x: C_1)$  or  $(x: C_2)$  or both are not in  $S$ .
- 2  $S \rightarrow_{\sqcup} \{x: D\} \cup S$   
if  $(x: C_1 \sqcup C_2) \in S$  and neither  $(x: C_1) \in S$  nor  $(x: C_2) \in S$  and  $D = C_1$  *or*  $D = C_2$ .
- 3  $S \rightarrow_{\exists} \{xry, y: C\} \cup S$   
if  $(x: \exists r.C) \in S$ ,  $y$  is a *fresh variable*, and there is no  $z$  s.t.  $(xrz) \in S$  and  $(z: C) \in S$ .
- 4  $S \rightarrow_{\forall} \{y: C\} \cup S$   
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Deterministic rules (1,3,4) vs. non-deterministic (2).

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# Tableau Method (4): Invariances

## Theorem (Invariance)

*Let  $S$  and  $T$  be constraint systems:*

- 1 If  $T$  has been generated by applying a deterministic rule to  $S$ , then  $S$  is satisfiable iff  $T$  is satisfiable.*
- 2 If  $T$  has been generated by applying a non-deterministic rule to  $S$ , then  $S$  is satisfiable if  $T$  is satisfiable. Furthermore, if a non-deterministic rule can be applied to  $S$ , then it can be applied such that  $S$  is satisfiable iff the resulting system  $T$  is satisfiable.*

## Theorem (Termination)

*Let  $C$  be an  $\mathcal{ALC}$  concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system  $\{x: C\}$ .*

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## Theorem (Invariance)

Let  $S$  and  $T$  be constraint systems:

- 1 If  $T$  has been generated by applying a deterministic rule to  $S$ , then  $S$  is satisfiable iff  $T$  is satisfiable.
- 2 If  $T$  has been generated by applying a non-deterministic rule to  $S$ , then  $S$  is satisfiable if  $T$  is satisfiable.  
Furthermore, if a non-deterministic rule can be applied to  $S$ , then it can be applied such that  $S$  is satisfiable iff the resulting system  $T$  is satisfiable.

## Theorem (Termination)

Let  $C$  be an  $\mathcal{ALC}$  concept description in NNF. Then there exists no infinite chain of transformations starting from the constraint system  $\{x: C\}$ .

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# Tableau Method (5): Soundness and Completeness

A constraint system is called **closed** if no transformation rule can be applied.

A **clash** is a pair of constraints of the form  $x : A$  and  $x : \neg A$ , where  $A$  is a concept name.

Theorem (Soundness and Completeness)

*A closed constraint system is satisfiable iff it does not contain a clash.*

Proof idea.

$\Rightarrow$ : obvious.  $\Leftarrow$ : Construct a model by using the concept labels.

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# Space Requirements

Because the tableau method is *non-deterministic* ( $\rightarrow_{\square}$  rule) ... there could be exponentially many closed constraint systems in the end.

Interestingly, even one constraint system can have *exponential size*.

**Example:**

$$\begin{array}{l} \exists r. A \sqcap \exists r. B \sqcap \\ \forall r. \left( \begin{array}{l} \exists r. A \sqcap \exists r. B \sqcap \\ \forall r. \left( \begin{array}{l} \exists r. A \sqcap \exists r. B \sqcap \\ \forall r. (\dots) \end{array} \right) \end{array} \right) \end{array}$$

**However:** One can modify the algorithm so that it needs only poly. space.

**Idea:** Generating a  $y$  only for one  $\exists r. C$  and then proceeding into the depth.

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ABox Reasoning

ABox satisfiability can also be decided using the tableau method if we can add constraints of the form  $x \neq y$  (for *UNA*):

- *Normalize* and *unfold* and add inequalities for all pairs of objects mentioned in the ABox.
- Strictly speaking, in *ALC* we do not need this because we are never *forced* to identify two objects.

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