

Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning II: Allen's Interval Calculus

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Allen's Interval Calculus – Outline

- 1 Allen's Interval Calculus
 - Motivation
 - Intervals and Relations Between Them
 - Composing Interval Relations
- 2 Reasoning in Allen's Interval Calculus
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Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- **NLP** – we do not have precise time points
- **Planning** – we do not want to commit to time points too early
- **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- **Time intervals**: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?

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Motivation: Example

Consider a planning scenario for multimedia generation:

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point to* Plug-in-Picture1.

Temporal relations between events:

P2 should happen during P1

P3 should happen during P1

P2 should happen before or directly precede P3

P4 should happen during or end together with P2

⇒ P4 happens before or directly precedes P3

⇒ We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

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Allen's Interval Calculus

- Allen's interval calculus: **time intervals** and **binary relations** over them
- **Time intervals**: $X = (X^-, X^+)$, where X^- and X^+ are interpreted over the reals and $X^- < X^+$ (\rightsquigarrow naïve approach)
- **Relations** between concrete intervals, e. g.:

(1.0,2.0) strictly before (3.0,5.5)

(1.0,3.0) meets (3.0,5.5)

(1.0,4.0) overlaps (3.0,5.5)

...

\rightsquigarrow Which relations are conceivable?

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The Base Relations

How many ways are there to order the four points of two intervals?

| Relation | Symbol | Name |
|--------------------------------------|----------|----------|
| $\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$ | \prec | before |
| $\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$ | m | meets |
| $\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$ | o | overlaps |
| $\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$ | s | starts |
| $\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$ | f | finishes |
| $\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$ | d | during |
| $\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$ | \equiv | equal |

and the **converse** relations (obtained by exchanging X and Y)

↪ These relations are JEPD.

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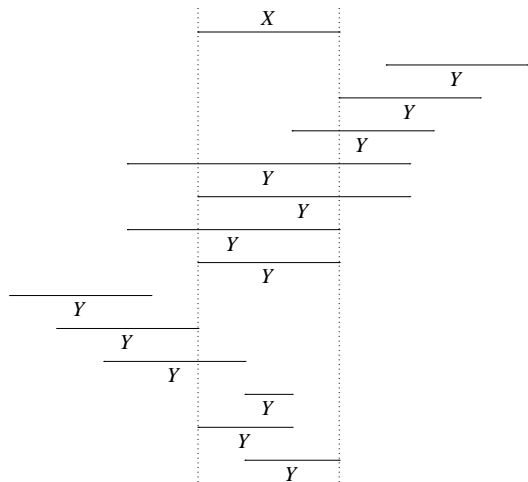
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The 13 Base Relations Graphically



before
meets
overlaps
during
starts
finishes
equals
before⁻¹
meets⁻¹
overlaps⁻¹
during⁻¹
starts⁻¹
finishes⁻¹

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Disjunctive Descriptions

- Assumption: We don't have precise information about the relation between X and Y , e. g.:

$$X \circ Y \text{ or } X \text{ m } Y$$

- ... modelled by sets of base relations (meaning the union of the relations):

$$X \{o, m\} Y$$

↪ 2^{13} imprecise relations (incl. \emptyset and \mathbf{B})

Example of an indefinite qualitative description:

$$\left\{ X \{o, m\} Y, Y \{m\} Z, X \{o, m\} Z \right\}$$

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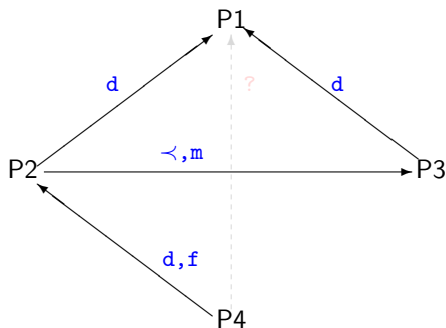
Our Example. . . Formally

P1: *Display Picture1*

P3: *Say "The device should be shut off."*

P2: *Say "Put the plug in."*

P4: *Point to Plug-in-Picture1.*



Compose the constraints: $P4 \{d, f\} P2$ and $P2 \{d\} P1$
 $\rightsquigarrow P4 \{d\} P1$.

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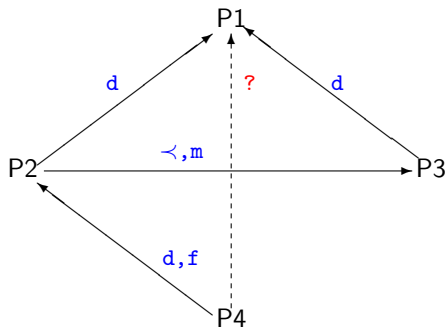
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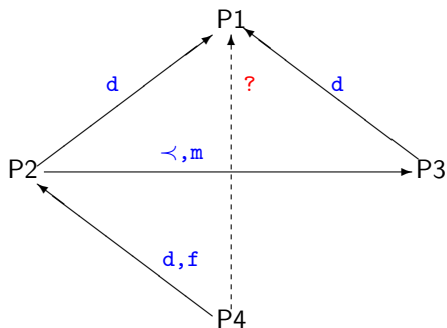
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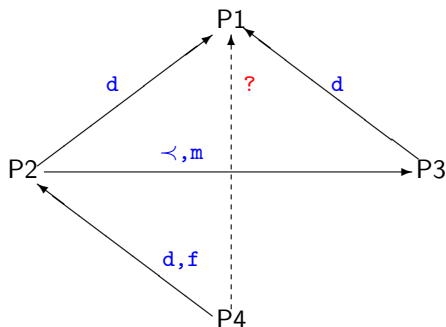
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Composition of Base Relations

| | λ | γ | δ | δ^{-1} | \circ | \circ^{-1} | \exists | \exists^{-1} | s | s^{-1} | $\#$ | $\#^{-1}$ |
|----------------|--------------------------------|-------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|--------------------------------|
| λ | λ | B | $\exists \lambda \delta \circ$ | λ | λ | $\exists \lambda \delta \circ$ | λ | $\exists \lambda \delta \circ$ | λ | λ | $\exists \lambda \delta \circ$ | λ |
| γ | B | γ | $\exists \gamma \delta \circ$ | γ | $\exists \gamma \delta \circ$ | γ | $\exists \gamma \delta \circ$ | γ | $\exists \gamma \delta \circ$ | γ | γ | γ |
| δ | λ | γ | δ | B | $\exists \lambda \delta \circ$ | $\exists \gamma \delta \circ$ | λ | γ | δ | $\exists \gamma \delta \circ$ | δ | $\exists \lambda \delta \circ$ |
| δ^{-1} | $\exists \lambda \delta \circ$ | $\exists \gamma \delta \circ$ | $\exists \lambda \delta \circ$ | δ^{-1} | δ^{-1} | $\exists \lambda \delta \circ$ | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} |
| \circ | λ | $\exists \gamma \delta \circ$ | $\exists \lambda \delta \circ$ | δ^{-1} | δ^{-1} | $\exists \lambda \delta \circ$ | λ | \circ | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} |
| \circ^{-1} | $\exists \lambda \delta \circ$ | γ | δ^{-1} | $\exists \gamma \delta \circ$ | B | $\exists \lambda \delta \circ$ | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} |
| \exists | λ | $\exists \gamma \delta \circ$ | δ^{-1} | λ | λ | δ^{-1} | λ | $\exists \lambda \delta \circ$ | \exists | \exists | δ^{-1} | λ |
| \exists^{-1} | $\exists \lambda \delta \circ$ | γ | δ^{-1} | γ | δ^{-1} | γ | $\exists \lambda \delta \circ$ | γ | δ^{-1} | γ | $\exists \lambda \delta \circ$ | $\exists \lambda \delta \circ$ |
| s | λ | γ | δ | $\exists \lambda \delta \circ$ | δ^{-1} | δ^{-1} | λ | $\exists \lambda \delta \circ$ | s | $\exists \lambda \delta \circ$ | δ | $\exists \lambda \delta \circ$ |
| s^{-1} | $\exists \lambda \delta \circ$ | γ | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} | $\exists \lambda \delta \circ$ | $\exists \lambda \delta \circ$ | $\exists \lambda \delta \circ$ | δ^{-1} | δ^{-1} |
| $\#$ | λ | γ | δ | $\exists \gamma \delta \circ$ | δ^{-1} | δ^{-1} | \exists | δ | $\exists \lambda \delta \circ$ | δ | δ | δ |
| $\#^{-1}$ | λ | $\exists \gamma \delta \circ$ | δ^{-1} | δ^{-1} | δ^{-1} | δ^{-1} | \exists | δ | δ^{-1} | δ^{-1} | δ | δ |

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- Using the **composition table** and the rules about operations on relations, we can **deduce** new relations between time intervals.
- What would be a **systematic** approach?
- How costly is that?
- Is that **complete**?
- If not, could it be complete on a subset of the relation system?

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Constraint Propagation – The Naive Algorithm

Enforcing path consistency using the straight-forward method:
Let $Table[i, j]$ be an array of size $n \times n$ (n : number of intervals) in which we record the constraints between the intervals.

EnforcePathConsistency1(\mathcal{C})

Input: a (binary) CSP $\mathcal{C} = \langle V, D, \mathcal{C} \rangle$

Output: an equivalent, but path consistent CSP \mathcal{C}'

repeat

for each pair (i, j) , $1 \leq i, j \leq n$

for each k with $1 \leq k \leq n$

$Table[i, j] := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

until no entry in $Table$ is changed

↪ terminates;

↪ needs $O(n^5)$ intersections and compositions.

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An $O(n^3)$ Algorithm

EnforcePathConsistency2(\mathcal{C})

Input: a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$

Output: an equivalent, but path consistent CSP \mathcal{C}'

$Paths(i, j) = \{(i, j, k) : 1 \leq k \leq n\} \cup \{(k, i, j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i,j} Paths(i, j)$

while $Q \neq \emptyset$

 select and delete (i, k, j) from Q

$T := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

if $T \neq Table[i, j]$

$Table[i, j] := T$

$Table[j, i] := T^{-1}$

$Queue := Queue \cup Paths(i, j)$

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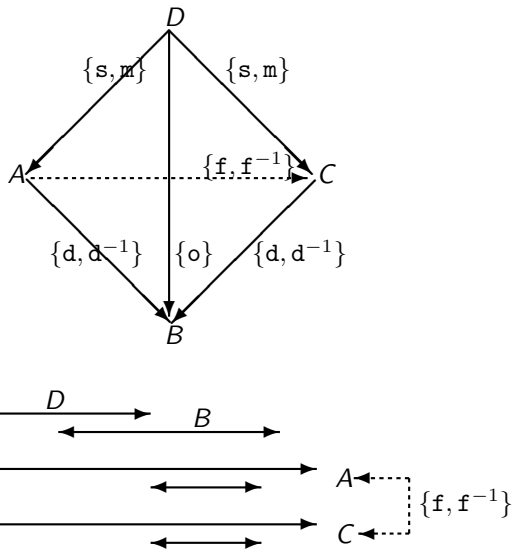
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Example for Incompleteness



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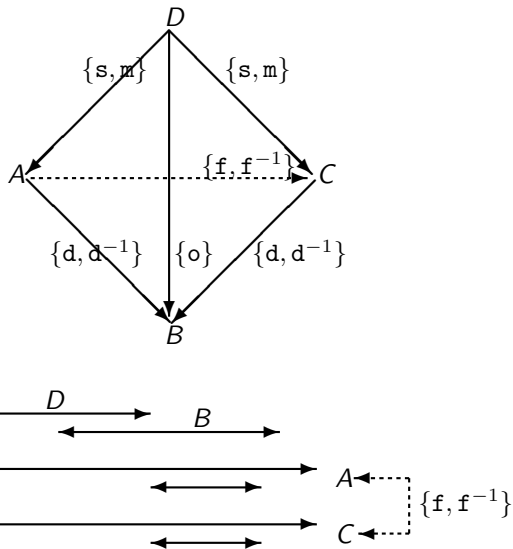
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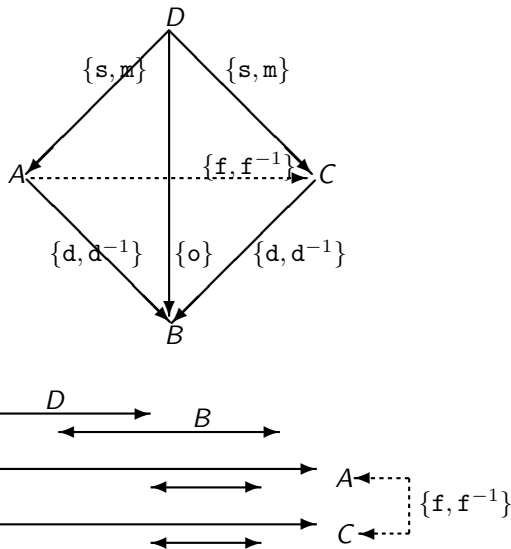
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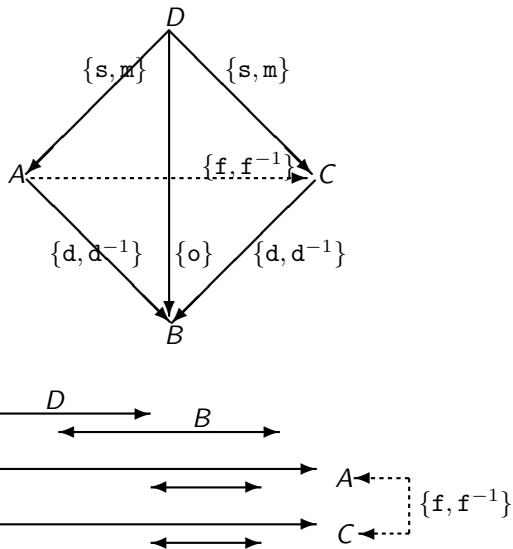
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NP-Hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.

Reduction from **3-colorability** (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \dots, v_n\}$ be an instance of 3-colorability.

Then we use the intervals $\{v_1, \dots, v_n, 1, 2, 3\}$ with the following constraints:

$$\begin{array}{lll} 1 & \{m\} & 2 \\ 2 & \{m\} & 3 \\ v_i & \{m, \equiv, m^{-1}\} & 2 \quad \forall v_i \in V \\ v_i & \{m, m^{-1}, \prec, \succ\} & v_j \quad \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable *iff* G can be colored with 3 colors. □

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Looking for Special Cases

- **Idea:** Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- **Note:** Interval formulae $X R Y$ can be expressed as **clauses** over **atoms** of the form $a \text{ op } b$, where:
 - a and b are endpoints X^-, X^+, Y^- and Y^+ and
 - $\text{op} \in \{<, >, =, \leq, \geq\}$.
- **Example:** All base relations can be expressed as unit clauses.

Lemma

Let $\pi(\Theta)$ be the translation of Θ to clause form. Θ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

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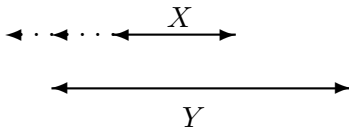
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The Continuous Endpoint Class

Continuous Endpoint Class \mathcal{C} : This is a subset of \mathcal{A} such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is **forbidden**.

Example: All basic relations and $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \{X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, \\ X^+ < Y^+\}$$



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Why Do We Have Completeness?

The set \mathcal{C} is **closed** under intersection, composition, and converse (it is a **sub-algebra** wrt. these three operations on relations). This can be shown by using a computer program.

Lemma

Each 3-consistent interval CSP over \mathcal{C} is globally consistent.

Theorem (van Beek)

Path consistency solves $\text{CMIN}(\mathcal{C})$ and decides $\text{CSAT}(\mathcal{C})$.

(Proof: Follows from the above lemma and the fact that a strongly n -consistent CSP is minimal.)

Corollary

A path consistent interval CSP consisting of base relations only is satisfiable.

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Helly's Theorem

Definition

A set $M \subseteq \mathbf{R}^n$ is **convex** iff for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M .

Theorem (Helly)

Let F be a finite family of at least $n + 1$ convex sets in \mathbf{R}^n . If all sub-families of F with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

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Strong n -Consistency (1)

Proof (Part 1).

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3$ ✓

Induction assumption: Assume strong $(k - 1)$ -consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k - 1$ variables.

We have to show that we can extend the instantiation to any k th variable.

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Induction assumption: Assume strong $(k - 1)$ -consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k - 1$ variables.

We have to show that we can extend the instantiation to any k th variable.

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Strong n -Consistency (1)

Proof (Part 1).

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Strong n -Consistency (2): Instantiating the k th Variable

Proof (Part 2).

The instantiation of the $k - 1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k .

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \dots$$

Such inequalities define convex subsets in \mathbf{R}^2 .

\rightsquigarrow Consider sets of 3 inequalities (= 3 convex sets).

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Strong n -Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 2 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+ , but do not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by 3-consistency there exists a common point.

↪ With Helly's Theorem, there exists an instantiation consistent with **all** inequalities.

↪ Strong k -consistency for all $k \leq n$. □

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Outlook

- $\text{CMIN}(\mathcal{C})$ can be computed in $O(n^3)$ time (for n being the number of intervals) using the path consistency algorithm.
- \mathcal{C} is a set of relations occurring “naturally” when observations are uncertain.
- \mathcal{C} contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? **Probably not.**
- Are there larger sets of relations that permit polynomial satisfiability testing? **Yes.**

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- 3 A Maximal Tractable Sub-Algebra**
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Literature

The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only **unit** clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(X \{d, o\} Y) = \{X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ X^+ < Y^+\}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

Enforcing path consistency decides CSAT(\mathcal{P}).

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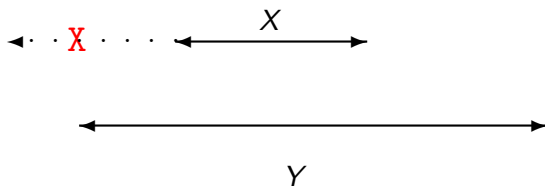
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Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

Enforcing path consistency decides CSAT(P).

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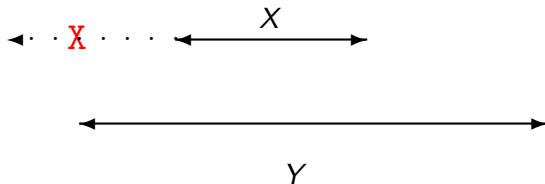
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The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only **Horn clauses** where only the following **literals** are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{o, s, f^{-1}\}$:

$$\pi(X\{o, s, f^{-1}\}Y) = \left\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \right\}.$$

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Partial Orders: The *ORD* Theory

Let *ORD* be the following theory:

$$\forall x, y, z: \quad x \leq y \wedge y \leq z \quad \rightarrow \quad x \leq z \quad (\textit{transitivity})$$

$$\forall x: \quad x \leq x \quad (\textit{reflexivity})$$

$$\forall x, y: \quad x \leq y \wedge y \leq x \quad \rightarrow \quad x = y \quad (\textit{anti-symmetry})$$

$$\forall x, y: \quad x = y \quad \rightarrow \quad x \leq y \quad (\textit{weakening of =})$$

$$\forall x, y: \quad x = y \quad \rightarrow \quad y \leq x \quad (\textit{weakening of =}).$$

- *ORD* describes partially ordered sets, \leq being the ordering relation.
- *ORD* is a **Horn theory**
- What is missing wrt. *dense* and *linear* orders?

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Satisfiability over Partial Orders

Proposition

Let Θ be a CSP over \mathcal{H} . Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

\Rightarrow : Since the reals form a partially ordered set (i. e., satisfy ORD), this direction is trivial.

\Leftarrow : Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. □

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Complexity of CSAT(\mathcal{H})

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

Theorem

$CSAT(\mathcal{H})$ can be decided in polynomial time.

Proof.

$CSAT(\mathcal{H})$ instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. \square

$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H}$ with $|\mathcal{C}| = 83$, $|\mathcal{P}| = 188$, $|\mathcal{H}| = 868$

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Path Consistency and the OH-Class

Lemma

Let Θ be a path-consistent set over \mathcal{H} . Then

$$(X \{ \} Y) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

Proof idea: One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt. **positive unit resolution**. Since this inference rule is refutation complete for Horn theories, the claim follows.

Theorem

Enforcing path consistency decides $CSAT(\mathcal{H})$.

- ↪ Maximality of \mathcal{H} ?
- ↪ Do we have to check all 8192 – 868 extensions?

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Complexity of Sub-Algebras

Let \hat{S} be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by S).

Theorem

CSAT(\hat{S}) can be polynomially transformed to CSAT(S).

Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in S , introducing perhaps some fresh variables. □

- ↪ Polynomiality of S extends to \hat{S} .
- ↪ NP-hardness of \hat{S} is inherited by all generating sets S .
- ↪ **Note:** $\mathcal{H} = \hat{\mathcal{H}}$.

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Minimal Extensions of the \mathcal{H} -Subclass

A **computer-aided** case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras that strictly contain \mathcal{H} : $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

Theorem

$CSAT(\mathcal{H} \cup \{N_i\})$ is NP-complete.

Question: Are there other **maximal** tractable subclasses?

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“Interesting” Subclasses

Interesting subclasses of \mathcal{A} should contain all basic relations.

A *computer-aided* case analysis reveals:

For $S \supseteq \{\{B\} : B \in \mathbf{B}\}$ it holds that

- 1 $\hat{S} \subseteq \mathcal{H}$, or
- 2 N_1 or N_2 is in \hat{S} .

In case 2, one can show: $\text{CSAT}(S)$ is NP-complete.

$\rightsquigarrow \mathcal{H}$ is the **only interesting** maximal tractable subclass.

If we include non-interesting subalgebras, there exist exactly 18 tractable classes.

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Relevance?

Theory: \oplus We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*.

Practice: \ominus All known applications either need only \mathcal{P} or they need more than \mathcal{H} !

Backtracking methods might profit from the result by **reducing the branching factor**.

- \rightsquigarrow How difficult is CSAT(\mathcal{A}) in practice?
- \rightsquigarrow What are the relevant branching factors?

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Solving General Allen CSPs

- Backtracking algorithm using **path consistency** as a **forward-checking method**
 - Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
 - Refinements and evaluation of different heuristics
- ↪ Which tractable fragment should one use?

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Branching Factors

- If the labels are split into **base relations**, then on average a label is split into

6.5 relations

- If the labels are split into **pointizable relations** (\mathcal{P}), then on average a label is split into

2.955 relations

- If the labels are split into **ORD-Horn relations** (\mathcal{H}), then on average a label is split into

2.533 relations

⇒ A difference of **0.422**

⇒ This makes a difference for “hard” instances.

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Summary

- Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the **continuous endpoint class**, minimal CSPs can be computed using the path-consistency method.
- For the larger **ORD-Horn class**, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.

KRR

Nebel,
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Wöfl

Allen's
Interval
Calculus

Reasoning in
Allen's
Interval
Calculus

A Maximal
Tractable
Sub-Algebra

The Endpoint
Subclass
The ORD-Horn
Subclass

Maximality
Solving Arbitrary
Allen CSPs

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