

Principles of Knowledge Representation and Reasoning

Qualitative Representation and Reasoning II: Allen's Interval Calculus

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Allen's Interval Calculus

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Intervals and Relations Between Them

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Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- ▶ **NLP** – we do not have precise time points
- ▶ **Planning** – we do not want to commit to time points too early
- ▶ **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- ▶ **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- ▶ **Time intervals**: actions and events have duration
- ▶ Reducibility? Expressiveness? Computational costs for reasoning?

Motivation: Example

Consider a planning scenario for multimedia generation:

- P1: *Display Picture1*
- P2: *Say "Put the plug in."*
- P3: *Say "The device should be shut off."*
- P4: *Point to Plug-in-Picture1.*

Temporal relations between events:

P2	should happen during	P1
P3	should happen during	P1
P2	should happen before or directly precede	P3
P4	should happen during or end together with	P2

- ↪ P4 happens before or directly precedes P3
- ↪ We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

Allen's Interval Calculus

- ▶ Allen's interval calculus: **time intervals** and **binary relations** over them
- ▶ **Time intervals**: $X = (X^-, X^+)$, where X^- and X^+ are interpreted over the reals and $X^- < X^+$ (\rightsquigarrow naive approach)
- ▶ **Relations** between concrete intervals, e. g.:
 - $(1.0, 2.0)$ strictly before $(3.0, 5.5)$
 - $(1.0, 3.0)$ meets $(3.0, 5.5)$
 - $(1.0, 4.0)$ overlaps $(3.0, 5.5)$
 - ...
- ↪ Which relations are conceivable?

The Base Relations

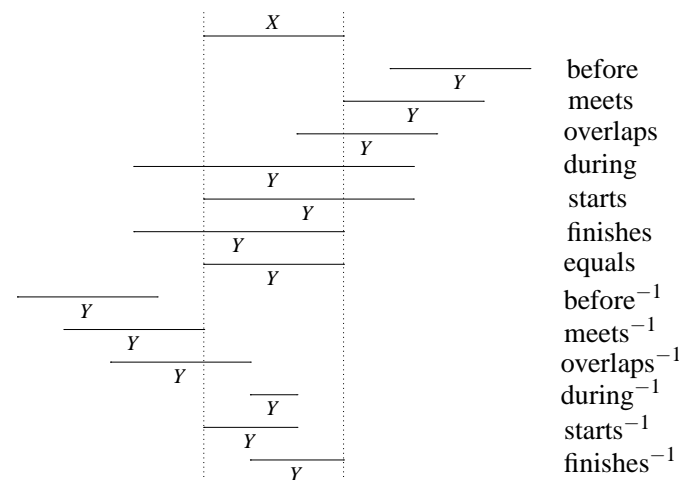
How many ways are there to order the four points of two intervals?

Relation	Symbol	Name
$\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$	$<$	before
$\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$	m	meets
$\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$	o	overlaps
$\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$	s	starts
$\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$	f	finishes
$\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$	d	during
$\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$	\equiv	equal

and the **converse** relations (obtained by exchanging X and Y)

- ↪ These relations are JEPD.

The 13 Base Relations Graphically



Reasoning in Allen's Interval Calculus

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Constraint Propagation – The Naive Algorithm

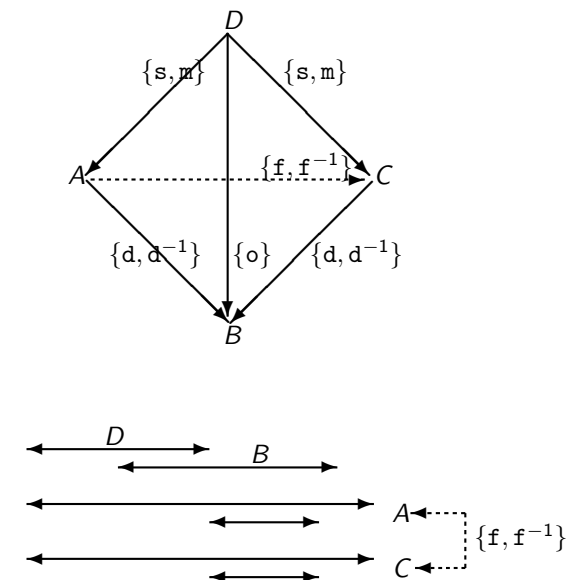
Enforcing path-consistency using the straight-forward method:

Let $Table[i, j]$ be an array of size $|n| \times |n|$ (n : number of intervals), in which we have recorded the constraints between the intervals.**EnforcePathConsistency1** (\mathcal{C}):*Input:* a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$ *Output:* an equivalent, but path consistent CSP \mathcal{C}' **repeat****for** each pair (i, j) , $1 \leq i, j \leq n$ **for** each k with $1 \leq k \leq n$ $Table[i, j] := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$ **endfor****endfor****until** no entry in $Table$ is changed

↪ terminates;

↪ needs $O(n^5)$ intersections and compositions.An $O(n^3)$ Algorithm**EnforcePathConsistency2** (\mathcal{C}):*Input:* a (binary) CSP $\mathcal{C} = \langle V, D, C \rangle$ *Output:* an equivalent, but path consistent CSP \mathcal{C}' $Paths(i, j) = \{(i, j, k) : 1 \leq k \leq n\} \cup \{(k, i, j) : 1 \leq k \leq n\}$ $Queue := \bigcup_{i, j} Paths(i, j)$ **While** $Q \neq \emptyset$ select and delete (i, k, j) from Q $T := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$ **if** $T \neq Table[i, j]$ $Table[i, j] := T$ $Table[j, i] := T^{-1}$ $Queue := Queue \cup Paths(i, j)$ **endif****endwhile**

Example for Incompleteness



NP-Hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen's interval calculus.

Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \dots, v_n\}$ be an instance of 3-colorability.

Then we use the intervals $\{v_1, \dots, v_n, 1, 2, 3\}$ with the following constraints:

$$\begin{array}{lll} 1 & \{m\} & 2 \\ 2 & \{m\} & 3 \\ v_i & \{m, \equiv, m^{-1}\} & 2 \quad \forall v_i \in V \\ v_i & \{m, m^{-1}, \prec, \succ\} & v_j \quad \forall (v_i, v_j) \in E \end{array}$$

This constraint system is satisfiable iff G can be colored with 3 colors. \square

Looking for Special Cases

- ▶ **Idea:** Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- ▶ **Note:** Interval formulae $X R Y$ can be expressed as clauses over atoms of the form $a \text{ op } b$, where:
 - ▶ a and b are endpoints X^-, X^+, Y^- and Y^+ and
 - ▶ $\text{op} \in \{<, >, =, \leq, \geq\}$.
- ▶ **Example:** All base relations can be expressed as unit clauses.

Lemma

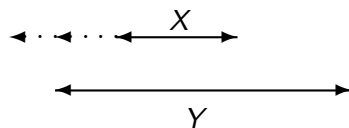
Let $\pi(\Theta)$ be the translation of Θ to clause form. Θ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

The Continuous Endpoint Class

Continuous Endpoint Class \mathcal{C} : This is a subset of \mathcal{A} such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and $\{d, o, s\}$, because

$$\pi(X \{d, o, s\} Y) = \begin{cases} X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, \\ X^+ < Y^+ \end{cases}$$



Why Do We Have Completeness?

The set \mathcal{C} is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

Lemma

Each 3-consistent interval CSP over \mathcal{C} is globally consistent.

Theorem (van Beek)

Path consistency solves $\text{CMIN}(\mathcal{C})$ and decides $\text{CSAT}(\mathcal{C})$.

Proof.

Follows from the above lemma and the fact that a strongly n -consistent CSP is minimal. \square

Corollary

A path consistent interval CSP consisting of base relations only is satisfiable.

Helly's Theorem

Definition

A set $M \subseteq \mathbf{R}^n$ is **convex** iff for all pairs of points $a, b \in M$, all points on the line connecting a and b belong to M .

Theorem (Helly)

Let F be a family of at least $n + 1$ convex sets in \mathbf{R}^n . If all sub-families of F with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

Strong n -Consistency (1)

Proof.

We prove the claim by induction over k with $k \leq n$.

Base case: $k = 1, 2, 3$ \checkmark

Induction assumption: Assume strong $k - 1$ -consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables X_i to pairs (s_i, e_i) satisfying the constraints R_{ij} between the $k - 1$ variables.

We have to show that we can extend the instantiation to any k th variable.

Strong n -Consistency (2): Instantiating the k th Variable

Proof (Part 2).

The instantiation of the $k - 1$ variables X_i to (s_i, e_i) restricts the instantiation of X_k .

Note: Since $R_{ij} \in \mathcal{C}$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \dots$$

Such inequalities define convex subsets in \mathbf{R}^2 .

\rightsquigarrow Consider sets of 3 inequalities (= 3 convex sets).

Strong n -Consistency (3): Using Helly's Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only X_k^- (or mention only X_k^+). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

Case 2: The inequalities mention X_k^- and X_k^+ , but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. X_k) can be involved and by the same argument as above there exists a common point.

\rightsquigarrow With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.

\rightsquigarrow Strong k -consistency for all $k \leq n$.

Outlook

- ▶ $\text{CMIN}(\mathcal{C})$ can be computed in $O(n^3)$ time (for n being the number of intervals) using the path consistency algorithm.
- ▶ \mathcal{C} is a set of relations occurring “naturally” when observations are uncertain.
- ▶ \mathcal{C} contains 83 relations (incl. the impossible and the universal relations).
- ▶ Are there larger sets such that path consistency computes minimal CSPs? **Probably not**
- ▶ Are there larger sets of relations that permit polynomial satisfiability testing? **Yes**

A Maximal Tractable Sub-Algebra

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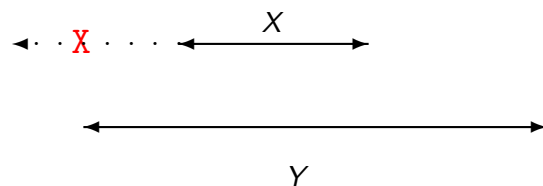
Literature

The EP-Subclass

End-Point Subclass: $\mathcal{P} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only **unit** clauses ($a \neq b$ is allowed).

Example: all basic relations and $\{d, o\}$ since

$$\pi(X \{d, o\} Y) = \{X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ X^+ < Y^+\}$$



Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

The path-consistency method decides $\text{CSAT}(\mathcal{P})$.

The ORD-Horn Subclass

ORD-Horn Subclass: $\mathcal{H} \subseteq \mathcal{A}$ is the subclass that permits a clause form containing only **Horn clauses**, where only the following **literals** are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$ is not allowed!

Example: all $R \in \mathcal{P}$ and $\{o, s, f^{-1}\}$:

$$\pi(X \{o, s, f^{-1}\} Y) = \left\{ X^- \leq X^+, X^- \neq X^+, \right. \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ \left. X^- \neq Y^- \vee X^+ \neq Y^+ \right\}.$$

Partial Orders: The ORD Theory

Let *ORD* be the following theory:

$$\begin{aligned} \forall x, y, z: \quad & x \leq y \wedge y \leq z \rightarrow x \leq z && \text{(transitivity)} \\ \forall x: \quad & x \leq x && \text{(reflexivity)} \\ \forall x, y: \quad & x \leq y \wedge y \leq x \rightarrow x = y && \text{(anti-symmetry)} \\ \forall x, y: \quad & x = y \rightarrow x \leq y && \text{(weakening of =)} \\ \forall x, y: \quad & x = y \rightarrow y \leq x && \text{(weakening of =)}. \end{aligned}$$

- ▶ *ORD* describes partially ordered sets, \leq being the ordering relation.
- ▶ *ORD* is a **Horn theory**
- ▶ What is missing wrt to *dense* and *linear* orders?

Satisfiability over Partial Orders

Proposition

Let Θ be a CSP over \mathcal{H} . Θ is satisfiable over interval interpretations iff $\pi(\Theta) \cup \text{ORD}$ is satisfiable over arbitrary interpretations.

Proof.

\Rightarrow : Since the reals form a partially ordered set (i. e., satisfy *ORD*), this direction is trivial.

\Leftarrow : Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order. \square

Complexity of CSAT(\mathcal{H})

Let $\text{ORD}_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$\text{ORD} \cup \pi(\Theta)$ is satisfiable iff $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion! \square

Theorem

$\text{CSAT}(\mathcal{H})$ can be decided in polynomial time.

Proof.

$\text{CSAT}(\mathcal{H})$ instances can be translated into a propositional Horn theory with blowup $O(n^3)$ according to the previous Prop., and such a theory is decidable in polynomial time. \square

$$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H} \quad \text{with} \quad |\mathcal{C}| = 83, |\mathcal{P}| = 188, |\mathcal{H}| = 868$$

Path-Consistency and the OH-Class

Lemma

Let Θ be a path-consistent set over \mathcal{H} . Then

$$(X \{ \} Y) \notin \Theta \quad \text{iff} \quad \Theta \text{ is satisfiable}$$

Proof Idea.

One can show that $\text{ORD}_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt **positive unit resolution**. Since this inference rule is refutation complete for Horn theories, the claim follows. \square

Lemma

\mathcal{H} is closed under intersection, composition, and conversion.

Theorem

The path-consistency method decides $\text{CSAT}(\mathcal{H})$.

\rightsquigarrow Maximality of \mathcal{H} ?

\rightsquigarrow Do we have to check all 8192 - 868 extensions?

Complexity of Sub-Algebras

Let \hat{S} be the closure of $S \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by S)

Theorem

$CSAT(\hat{S})$ can be polynomially transformed to $CSAT(S)$.

Proof Idea.

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in S , introducing perhaps some fresh variables. \square

- \rightsquigarrow Polynomiality of S extends to \hat{S} .
- \rightsquigarrow NP-hardness of \hat{S} is inherited by all generating sets S .
- \rightsquigarrow Note: $\mathcal{H} = \hat{\mathcal{H}}$.

Minimal Extensions of the \mathcal{H} -Subclass

A computer-aided case analysis leads to the following result:

Lemma

There are only two minimal sub-algebras that strictly contain \mathcal{H} : $\mathcal{X}_1, \mathcal{X}_2$

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

Theorem

$CSAT(\mathcal{H} \cup \{N_i\})$ is NP-complete.

Question: Are there other maximal tractable subclasses?

“Interesting” Subclasses

Interesting subclasses of \mathcal{A} should contain all basic relations.

A computer-aided case analysis reveals: For $S \supseteq \{\{B\} : B \in \mathbf{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. N_1 or N_2 is in \hat{S} .

In case 2, one can show: $CSAT(S)$ is NP-complete.

- \rightsquigarrow \mathcal{H} is the **only** maximal tractable subclass that is **interesting**.

Meanwhile, there is a **complete classification** of all sub-algebras containing at least one basic relation [IJCAI 2001] ... but the question for sub-algebras not containing a basic relation is open.

Relevance?

Theoretical:

We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*. \oplus

Practical: All known applications either need only \mathcal{P} or they need more than \mathcal{H} ! \ominus

Backtracking methods might profit from the result because the branching factor is lower. $?$

- \rightsquigarrow How difficult is $CSAT(\mathcal{A})$ in practice?
- \rightsquigarrow What are the relevant branching factors?

Solving General Allen CSPs

- ▶ Backtracking algorithm using **path-consistency** as a **forward-checking method**
 - ▶ Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
 - ▶ Refinements and evaluation of different heuristics
- ↪ Which tractable fragment should one use?

Branching Factors

- ▶ If the labels are split into **base relations**, then on average a label is split into
6.5 relations
 - ▶ If the labels are split into **pointizable relations** (\mathcal{P}), then on average a label is split into
2.955 relations
 - ▶ If the labels are split into **ORD-Horn relations** (\mathcal{H}), then on average a label is split into
2.533 relations
- ↪ A difference of **0.422**
- ↪ This makes a difference for “hard” instances.

Summary

- ▶ Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- ▶ The satisfiability problem for CSPs using the relations is NP-complete.
- ▶ For the **continuous endpoint class**, minimal CSPs can be computed using the path-consistency method.
- ▶ For the larger **ORD-Horn class**, CSAT is still decided by the path-consistency method.
- ▶ Can be used in practice for backtracking algorithms.

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