

# Principles of Knowledge Representation and Reasoning

## Qualitative Representation and Reasoning II: Allen's Interval Calculus

Bernhard Nebel, Malte Helmert and Stefan Wöfl

Albert-Ludwigs-Universität Freiburg

June 24, 2008

# Principles of Knowledge Representation and Reasoning

June 24, 2008 — Qualitative Representation and Reasoning II: Allen's Interval Calculus

## Allen's Interval Calculus

- Motivation

- Intervals and Relations Between Them

- Processing an Example

- Composition Table

- Outlook

## Reasoning in Allen's Interval Calculus

- Constraint propagation algorithms (enforcing path consistency)

- Example for Incompleteness

- NP-Hardness Example

- The Continuous Endpoint Class

- Completeness for the CEP Class

## A Maximal Tractable Sub-Algebra

- The Endpoint Subclass

- The ORD-Horn Subclass

- Maximality

- Solving Arbitrary Allen CSPs

# Allen's Interval Calculus – Outline

## Allen's Interval Calculus

Motivation

Intervals and Relations Between Them

Processing an Example

Composition Table

Outlook

## Reasoning in Allen's Interval Calculus

## A Maximal Tractable Sub-Algebra

## Literature

# Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- ▶ **NLP** – we do not have precise time points
- ▶ **Planning** – we do not want to commit to time points too early
- ▶ **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- ▶ **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- ▶ **Time intervals**: actions and events have duration
- ▶ Reducibility? Expressiveness? Computational costs for reasoning?

## Motivation: Example

Consider a planning scenario for multimedia generation:

P1: *Display* Picture1

P2: *Say* "Put the plug in."

P3: *Say* "The device should be shut off."

P4: *Point to* Plug-in-Picture1.

Temporal relations between events:

P2	should happen during	P1
P3	should happen during	P1
P2	should happen before or directly precede	P3
P4	should happen during or end together with	P2

↪ P4 happens before or directly precedes P3

↪ We could add the statement "P4 does not overlap with P3" without creating an inconsistency.

# Allen's Interval Calculus

- ▶ Allen's interval calculus: **time intervals** and **binary relations** over them
- ▶ **Time intervals**:  $X = (X^-, X^+)$ , where  $X^-$  and  $X^+$  are interpreted over the reals and  $X^- < X^+$  ( $\rightsquigarrow$  naïve approach)
- ▶ **Relations** between concrete intervals, e. g.:

*(1.0,2.0) strictly before (3.0,5.5)*

*(1.0,3.0) meets (3.0,5.5)*

*(1.0,4.0) overlaps (3.0,5.5)*

...

$\rightsquigarrow$  Which relations are conceivable?

## The Base Relations

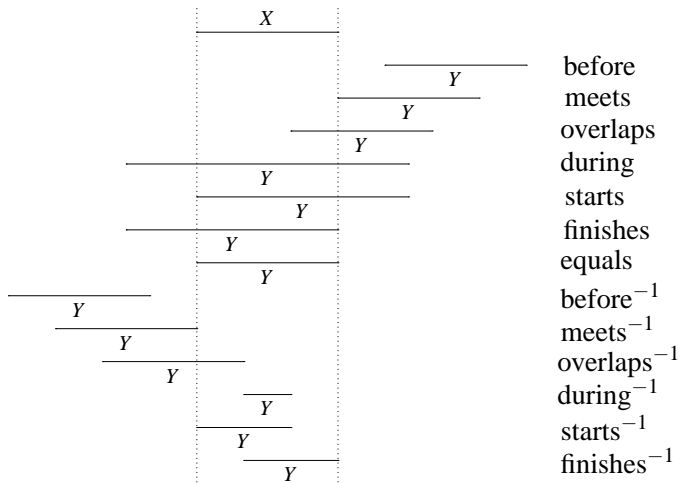
How many ways are there to order the four points of two intervals?

Relation	Symbol	Name
$\{(X, Y) : X^- < X^+ < Y^- < Y^+\}$	$\prec$	before
$\{(X, Y) : X^- < X^+ = Y^- < Y^+\}$	m	meets
$\{(X, Y) : X^- < Y^- < X^+ < Y^+\}$	o	overlaps
$\{(X, Y) : X^- = Y^- < X^+ < Y^+\}$	s	starts
$\{(X, Y) : Y^- < X^- < X^+ = Y^+\}$	f	finishes
$\{(X, Y) : Y^- < X^- < X^+ < Y^+\}$	d	during
$\{(X, Y) : Y^- = X^- < X^+ = Y^+\}$	$\equiv$	equal

and the **converse** relations (obtained by exchanging  $X$  and  $Y$ )

$\rightsquigarrow$  These relations are JEPD.

# The 13 Base Relations Graphically



## Disjunctive Descriptions

- ▶ Assumption: We don't have precise information about the relation between  $X$  and  $Y$ , e. g.:

$$X \circ Y \text{ or } X \text{ m } Y$$

- ▶ ... modelled by sets of base relations (meaning the union of the relations):

$$X \{o, m\} Y$$

↪  $2^{13}$  imprecise relations (incl.  $\emptyset$  and  $\mathbf{B}$ )

Example of an indefinite qualitative description:

$$\left\{ X \{o, m\} Y, Y \{m\} Z, X \{o, m\} Z \right\}$$

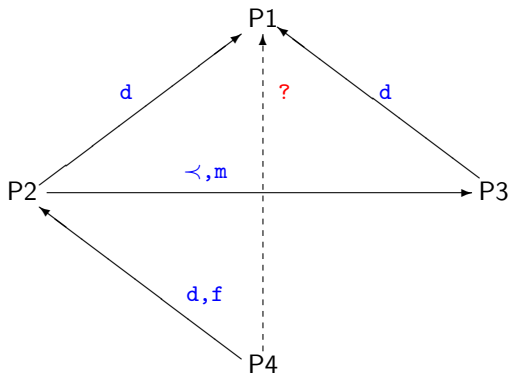
## Our Example ... Formal

P1: *Display* Picture1

P3: *Say* "The device should be shut off."

P2: *Say* "Put the plug in."

P4: *Point to* Plug-in-Picture1.



Compose the constraints:  $P4 \{d, f\} P2$  and  $P2 \{d\} P1$ :  $P4 \{d\} P1$ .



# Outlook

- ▶ Using the **composition table** and the rules about operations on relations, we can **deduce** new relations between time intervals.
- ▶ What would be a **systematic** approach?
- ▶ How costly is that?
- ▶ Is that **complete**?
- ▶ If not, could it be complete on a subset of the relation system?

# Reasoning in Allen's Interval Calculus

## Allen's Interval Calculus

### Reasoning in Allen's Interval Calculus

Constraint propagation algorithms (enforcing path consistency)

Example for Incompleteness

NP-Hardness Example

The Continuous Endpoint Class

Completeness for the CEP Class

## A Maximal Tractable Sub-Algebra

## Literature

## Constraint Propagation – The Naive Algorithm

Enforcing path-consistency using the straight-forward method:

Let  $Table[i, j]$  be an array of size  $|n| \times |n|$  ( $n$ : number of intervals), in which we have recorded the constraints between the intervals.

**EnforcePathConsistency1** ( $\mathcal{C}$ ):

*Input:* a (binary) CSP  $\mathcal{C} = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP  $\mathcal{C}'$

**repeat**

**for** each pair  $(i, j)$ ,  $1 \leq i, j \leq n$

**for** each  $k$  with  $1 \leq k \leq n$

$Table[i, j] := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

**endfor**

**endfor**

**until** no entry in  $Table$  is changed

↪ terminates;

↪ needs  $O(n^5)$  intersections and compositions.

## An $O(n^3)$ Algorithm

**EnforcePathConsistency2** ( $\mathcal{C}$ ):

*Input:* a (binary) CSP  $\mathcal{C} = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP  $\mathcal{C}'$

$Paths(i, j) = \{(i, j, k) : 1 \leq k \leq n\} \cup \{(k, i, j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i,j} Paths(i, j)$

**While**  $Q \neq \emptyset$

  select and delete  $(i, k, j)$  from  $Q$

$T := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

**if**  $T \neq Table[i, j]$

$Table[i, j] := T$

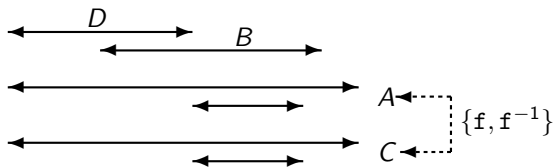
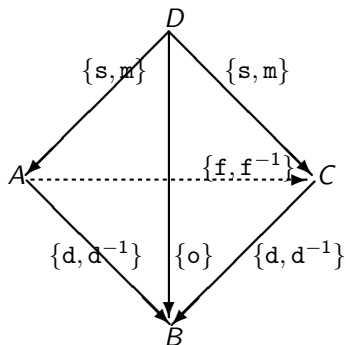
$Table[j, i] := T^{-1}$

$Queue := Queue \cup Paths(i, j)$

**endif**

**endwhile**

# Example for Incompleteness



# NP-Hardness

## Theorem (Kautz & Vilain)

*CSAT is NP-hard for Allen's interval calculus.*

### Proof.

Reduction from **3-colorability** (original proof using 3Sat).

Let  $G = (V, E)$ ,  $V = \{v_1, \dots, v_n\}$  be an instance of 3-colorability. Then we use the intervals  $\{v_1, \dots, v_n, 1, 2, 3\}$  with the following constraints:

$$\begin{array}{lll}
 1 & \{m\} & 2 \\
 2 & \{m\} & 3 \\
 v_i & \{m, \equiv, m^{-1}\} & 2 \quad \forall v_i \in V \\
 v_i & \{m, m^{-1}, \prec, \succ\} & v_j \quad \forall (v_i, v_j) \in E
 \end{array}$$

This constraint system is satisfiable *iff*  $G$  can be colored with 3 colors.  $\square$

## Looking for Special Cases

- ▶ **Idea:** Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.
- ▶ **Note:** Interval formulae  $X R Y$  can be expressed as **clauses** over **atoms** of the form  $a \text{ op } b$ , where:
  - ▶  $a$  and  $b$  are endpoints  $X^-, X^+, Y^-$  and  $Y^+$  and
  - ▶  $\text{op} \in \{<, >, =, \leq, \geq\}$ .
- ▶ **Example:** All base relations can be expressed as unit clauses.

### Lemma

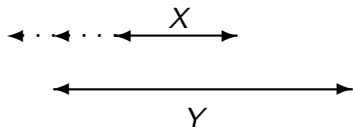
*Let  $\pi(\Theta)$  be the translation of  $\Theta$  to clause form.  $\Theta$  is satisfiable over intervals iff  $\pi(\Theta)$  is satisfiable over the rational numbers.*

# The Continuous Endpoint Class

**Continuous Endpoint Class  $\mathcal{C}$ :** This is a subset of  $\mathcal{A}$  such that there exists a clause form for each relation containing only unit clauses where  $\neg(a = b)$  is **forbidden**.

**Example:** All basic relations and  $\{d, o, s\}$ , because

$$\pi(X \{d, o, s\} Y) = \{X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, \\ X^+ < Y^+\}$$



## Why Do We Have Completeness?

The set  $\mathcal{C}$  is **closed** under intersection, composition, and converse (it is a **sub-algebra** wrt. these three operations on relations). This can be shown by using a computer program.

### Lemma

*Each 3-consistent interval CSP over  $\mathcal{C}$  is globally consistent.*

### Theorem (van Beek)

*Path consistency solves  $CMIN(\mathcal{C})$  and decides  $CSAT(\mathcal{C})$ .*

### Proof.

Follows from the above lemma and the fact that a strongly  $n$ -consistent CSP is minimal. □

### Corollary

*A path consistent interval CSP consisting of base relations only is satisfiable.*

# Helly's Theorem

## Definition

A set  $M \subseteq \mathbf{R}^n$  is **convex** iff for all pairs of points  $a, b \in M$ , all points on the line connecting  $a$  and  $b$  belong to  $M$ .

## Theorem (Helly)

*Let  $F$  be a family of at least  $n + 1$  convex sets in  $\mathbf{R}^n$ . If all sub-families of  $F$  with  $n + 1$  sets have a non-empty intersection, then  $\bigcap F \neq \emptyset$ .*

## Strong $n$ -Consistency (1)

Proof.

We prove the claim by induction over  $k$  with  $k \leq n$ .

**Base case:**  $k = 1, 2, 3$      $\checkmark$

**Induction assumption:** Assume strong  $k - 1$ -consistency (and non-emptiness of all relations)

**Induction step:** From the assumption, it follows that there is an instantiation of  $k - 1$  variables  $X_i$  to pairs  $(s_i, e_i)$  satisfying the constraints  $R_{ij}$  between the  $k - 1$  variables.

We have to show that we can extend the instantiation to any  $k$ th variable.

## Strong $n$ -Consistency (2): Instantiating the $k$ th Variable

### Proof (Part 2).

The instantiation of the  $k - 1$  variables  $X_i$  to  $(s_i, e_i)$  restricts the instantiation of  $X_k$ .

**Note:** Since  $R_{ij} \in \mathcal{C}$  by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \wedge e_j \geq X_k^- \wedge \dots$$

Such inequalities define convex subsets in  $\mathbf{R}^2$ .

$\rightsquigarrow$  Consider sets of 3 inequalities (= 3 convex sets).

## Strong $n$ -Consistency (3): Using Helly's Theorem

### Proof (Part 3).

**Case 1:** All 3 inequalities mention only  $X_k^-$  (or mention only  $X_k^+$ ). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

**Case 2:** The inequalities mention  $X_k^-$  and  $X_k^+$ , but it does not contain the inequality  $X_k^- < X_k^+$ . Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

**Case 3:** The set contains the inequality  $X_k^- < X_k^+$ . In this case, only three intervals (incl.  $X_k$ ) can be involved and by the same argument as above there exists a common point.

- ↪ With Helly's Theorem, it follows that there exists a consistent instantiation for all subsets of variables.
- ↪ Strong  $k$ -consistency for all  $k \leq n$ .

# Outlook

- ▶  $\text{CMIN}(\mathcal{C})$  can be computed in  $O(n^3)$  time (for  $n$  being the number of intervals) using the path consistency algorithm.
- ▶  $\mathcal{C}$  is a set of relations occurring “naturally” when observations are uncertain.
- ▶  $\mathcal{C}$  contains 83 relations (incl. the impossible and the universal relations).
- ▶ Are there larger sets such that path consistency computes minimal CSPs? **Probably not**
- ▶ Are there larger sets of relations that permit polynomial satisfiability testing? **Yes**

# A Maximal Tractable Sub-Algebra

Allen's Interval Calculus

Reasoning in Allen's Interval Calculus

## A Maximal Tractable Sub-Algebra

- The Endpoint Subclass

- The ORD-Horn Subclass

- Maximality

- Solving Arbitrary Allen CSPs

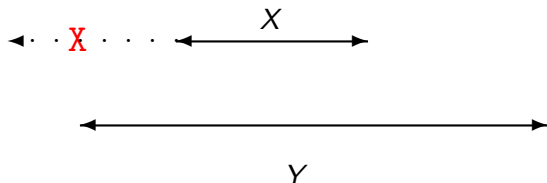
Literature

## The EP-Subclass

**End-Point Subclass:**  $\mathcal{P} \subseteq \mathcal{A}$  is the subclass that permits a clause form containing only **unit** clauses ( $a \neq b$  is allowed).

**Example:** all basic relations and  $\{d, o\}$  since

$$\pi(X \{d, o\} Y) = \{X^- < X^+, Y^- < Y^+, \\ X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\ X^+ < Y^+\}$$



**Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)**

*The path-consistency method decides CSAT( $\mathcal{P}$ ).*

## The ORD-Horn Subclass

**ORD-Horn Subclass:**  $\mathcal{H} \subseteq \mathcal{A}$  is the subclass that permits a clause form containing only **Horn clauses**, where only the following **literals** are allowed:

$$a \leq b, a = b, a \neq b$$

$\neg a \leq b$  is not allowed!

**Example:** all  $R \in \mathcal{P}$  and  $\{o, s, f^{-1}\}$ :

$$\pi(X\{o, s, f^{-1}\}Y) = \left\{ \begin{array}{l} X^- \leq X^+, X^- \neq X^+, \\ Y^- \leq Y^+, Y^- \neq Y^+, \\ X^- \leq Y^-, \\ X^- \leq Y^+, X^- \neq Y^+, \\ Y^- \leq X^+, X^+ \neq Y^-, \\ X^+ \leq Y^+, \\ X^- \neq Y^- \vee X^+ \neq Y^+ \end{array} \right\}.$$

# Partial Orders: The *ORD* Theory

Let *ORD* be the following theory:

$$\forall x, y, z: \quad x \leq y \wedge y \leq z \rightarrow x \leq z \quad (\textit{transitivity})$$

$$\forall x: \quad x \leq x \quad (\textit{reflexivity})$$

$$\forall x, y: \quad x \leq y \wedge y \leq x \rightarrow x = y \quad (\textit{anti-symmetry})$$

$$\forall x, y: \quad x = y \rightarrow x \leq y \quad (\textit{weakening of } =)$$

$$\forall x, y: \quad x = y \rightarrow y \leq x \quad (\textit{weakening of } =).$$

- ▶ *ORD* describes partially ordered sets,  $\leq$  being the ordering relation.
- ▶ *ORD* is a [Horn theory](#)
- ▶ What is missing wrt to *dense* and *linear* orders?

# Satisfiability over Partial Orders

## Proposition

Let  $\Theta$  be a CSP over  $\mathcal{H}$ .  $\Theta$  is satisfiable over interval interpretations iff  $\pi(\Theta) \cup ORD$  is satisfiable over arbitrary interpretations.

## Proof.

$\Rightarrow$ : Since the reals form a partially ordered set (i. e., satisfy  $ORD$ ), this direction is trivial.

$\Leftarrow$ : Each extension of a partial order to a linear order satisfies all formulae of the form  $a \leq b$ ,  $a = b$ , and  $a \neq b$  which have been satisfied over the original partial order. □

## Complexity of CSAT( $\mathcal{H}$ )

Let  $ORD_{\pi(\Theta)}$  be the propositional theory resulting from instantiating all axioms with the endpoints occurring in  $\pi(\Theta)$ .

### Proposition

$ORD \cup \pi(\Theta)$  is satisfiable iff  $ORD_{\pi(\Theta)} \cup \pi(\Theta)$  is so.

Proof idea: Herbrand expansion!



### Theorem

$CSAT(\mathcal{H})$  can be decided in polynomial time.

### Proof.

$CSAT(\mathcal{H})$  instances can be translated into a propositional Horn theory with blowup  $O(n^3)$  according to the previous Prop., and such a theory is decidable in polynomial time.



$$\mathcal{C} \subset \mathcal{P} \subset \mathcal{H} \quad \text{with} \quad |\mathcal{C}| = 83, |\mathcal{P}| = 188, |\mathcal{H}| = 868$$

# Path-Consistency and the OH-Class

## Lemma

Let  $\Theta$  be a path-consistent set over  $\mathcal{H}$ . Then

$$(X\{Y) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

## Proof Idea.

One can show that  $ORD_{\pi(\Theta)} \cup \pi(\Theta)$  is closed wrt **positive unit resolution**. Since this inference rule is refutation complete for Horn theories, the claim follows. □

## Lemma

$\mathcal{H}$  is closed under intersection, composition, and conversion.

## Theorem

The path-consistency method decides  $CSAT(\mathcal{H})$ .

↪ Maximality of  $\mathcal{H}$ ?

↪ Do we have to check all 8192 - 868 extensions?

## Complexity of Sub-Algebras

Let  $\hat{S}$  be the closure of  $S \subseteq \mathcal{A}$  under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by  $S$ )

### Theorem

$CSAT(\hat{S})$  can be polynomially transformed to  $CSAT(S)$ .

### Proof Idea.

All relations in  $\hat{S} - S$  can be modeled by a fixed number of compositions, intersections, and conversions of relations in  $S$ , introducing perhaps some fresh variables. □

- ↪ Polynomiality of  $S$  extends to  $\hat{S}$ .
- ↪ NP-hardness of  $\hat{S}$  is inherited by all generating sets  $S$ .
- ↪ **Note:**  $\mathcal{H} = \hat{\mathcal{H}}$ .

## Minimal Extensions of the $\mathcal{H}$ -Subclass

A *computer-aided* case analysis leads to the following result:

### Lemma

*There are only two minimal sub-algebras that strictly contain  $\mathcal{H}$ :  $\mathcal{X}_1, \mathcal{X}_2$*

$$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$$

$$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$$

The clause form of these relations contain “proper” disjunctions!

### Theorem

*CSAT( $\mathcal{H} \cup \{N_i\}$ ) is NP-complete.*

**Question:** Are there other **maximal** tractable subclasses?

## “Interesting” Subclasses

Interesting subclasses of  $\mathcal{A}$  should contain all basic relations.

A computer-aided case analysis reveals: For  $S \supseteq \{\{B\} : B \in \mathbf{B}\}$  it holds that

1.  $\hat{S} \subseteq \mathcal{H}$ , or
2.  $N_1$  or  $N_2$  is in  $\hat{S}$ .

In case 2, one can show:  $\text{CSAT}(S)$  is NP-complete.

$\rightsquigarrow \mathcal{H}$  is the **only** maximal tractable subclass that is **interesting**.

Meanwhile, there is a **complete classification** of all sub-algebras containing at least one basic relation [IJCAI 2001] ... but the question for sub-algebras not containing a basic relation is open.

# Relevance?

## Theoretical:

We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*. ⊕

**Practical:** All known applications either need only  $\mathcal{P}$  or they need more than  $\mathcal{H}$ ! ⊖

Backtracking methods might profit from the result because the branching factor is lower. ?

↪ How difficult is  $\text{CSAT}(\mathcal{A})$  in practice?

↪ What are the relevant branching factors?

# Solving General Allen CSPs

- ▶ Backtracking algorithm using **path-consistency** as a **forward-checking method**
- ▶ Relies on tractable fragments of Allen's calculus: split relations into relations of a tractable fragment, and backtrack over these.
- ▶ Refinements and evaluation of different heuristics
- ↪ Which tractable fragment should one use?

# Branching Factors

- ▶ If the labels are split into **base relations**, then on average a label is split into

**6.5 relations**

- ▶ If the labels are split into **pointizable relations** ( $\mathcal{P}$ ), then on average a label is split into

**2.955 relations**

- ▶ If the labels are split into **ORD-Horn relations** ( $\mathcal{H}$ ), then on average a label is split into

**2.533 relations**

↪ A difference of **0.422**

↪ This makes a difference for “hard” instances.

# Summary

- ▶ Allen's interval calculus is often adequate for describing relative orders of events that have duration.
- ▶ The satisfiability problem for CSPs using the relations is NP-complete.
- ▶ For the [continuous endpoint class](#), minimal CSPs can be computed using the path-consistency method.
- ▶ For the larger [ORD-Horn class](#), CSAT is still decided by the path-consistency method.
- ▶ Can be used in practice for backtracking algorithms.

# Literature I

american article

J.F. Allen.

Maintaining knowledge about temporal intervals.

*Communications of the ACM*, 26(11):832–843, November 1983.

Also in *Readings in Knowledge Representation*.

american article

P. van Beek and R. Cohen.

Exact and approximate reasoning about temporal relations.

*Computational Intelligence*, 6:132–144, 1990.

american article

B. Nebel and H.-J. Bürckert.

Reasoning about temporal relations: A maximal tractable subclass of Allen's interval algebra,

*Journal of the ACM*, 42(1): 43-66, 1995.

american article

B. Nebel.

Solving hard qualitative temporal reasoning problems: Evaluating the efficiency of using the ORD-horn class.

*CONSTRAINTS*, 1(3): 175-190, 1997.

# Literature II

eamericonanicle  A. Krokhin, P. Jeavons and P. Jonsson.

A complete classification of complexity in Allen's algebra in the presence of a non-trivial basic relation.

*Proc. 17th Int. Joint Conf. on AI (IJCAI-01), 83-88, Seattle, WA, 2001.*