

Principles of Knowledge Representation and Reasoning

Nonmonotonic Reasoning II: Minimal Models and Nonmonotonic Logic Programs

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Minimal Models and Nonmonotonic Logic Programs

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Minimal Model Reasoning

- ▶ Conflicts between defaults in default logic lead to multiple extensions
- ▶ Each extension corresponds to a maximal set of non-violated defaults
- ▶ Reasoning with defaults can also be achieved by a simpler mechanism: predicate or propositional logic + minimize the number of cases where a default (expressed as a conventional formula) is violated
⇒ **minimal models**
- ▶ Notion of **minimality**: cardinality vs. set-inclusion

Entailment with respect to Minimal Models

Definition

Let A be a set of atomic propositions. Let Φ be a set of propositional formulae on A , and $B \subseteq A$ a set (called **abnormalities**).

Then $\Phi \models_B \psi$ (ψ **B -minimally follows from Φ**) if $\mathcal{I} \models \psi$ for all interpretations \mathcal{I} such that $\mathcal{I} \models \Phi$ and there is no \mathcal{I}' such that $\mathcal{I}' \models \Phi$ and $\{b \in B \mid \mathcal{I}' \models b\} \subsetneq \{b \in B \mid \mathcal{I} \models b\}$.

Minimal models: example

$$\Phi = \left\{ \begin{array}{ll} \text{student} \wedge \neg \text{ABstudent} \rightarrow \neg \text{earnsmoney}, & \text{student,} \\ \text{adult} \wedge \neg \text{ABadult} \rightarrow \text{earnsmoney}, & \text{student} \rightarrow \text{adult} \end{array} \right\}$$

Φ has the following models.

- $\mathcal{I}_1 \models \text{student} \wedge \text{adult} \wedge \text{earnsmoney} \wedge \text{ABstudent} \wedge \text{ABadult}$
- $\mathcal{I}_2 \models \text{student} \wedge \text{adult} \wedge \neg \text{earnsmoney} \wedge \text{ABstudent} \wedge \text{ABadult}$
- $\mathcal{I}_3 \models \text{student} \wedge \text{adult} \wedge \text{earnsmoney} \wedge \text{ABstudent} \wedge \neg \text{ABadult}$
- $\mathcal{I}_4 \models \text{student} \wedge \text{adult} \wedge \neg \text{earnsmoney} \wedge \neg \text{ABstudent} \wedge \text{ABadult}$

Relation to Default Logic

We can embed propositional minimal model reasoning in the propositional default logic.

Theorem

Let A be a set of atomic propositions. Let Φ be a set of propositional formulae on A , and $B \subseteq A$.

Then $\Phi \models_B \psi$ if and only if ψ follows from $\langle D, W \rangle$ skeptically, where

$$D = \left\{ \frac{:\neg b}{\neg b} \mid b \in B \right\} \text{ and } W = \Phi.$$

Relation to Default Logic: Proof

Proof sketch.

“ \Rightarrow ”: Assume there is extension E of $\langle D, W \rangle$ such that $\psi \notin E$. Hence there is an interpretation \mathcal{I} such that $\mathcal{I} \models E$ and $\mathcal{I} \models \neg\psi$.

By the fact that there is no extension F such that $E \subset F$, \mathcal{I} is a B -minimal model of Φ . Hence ψ does not B -minimally follow from Φ .

“ \Leftarrow ”: Assume ψ does not B -minimally follow from Φ . Hence there is an B -minimal model \mathcal{I} of Φ such that $\mathcal{I} \not\models \psi$. Define

$$E = \text{Th}(\Phi \cup \{\neg b \mid b \in B, \mathcal{I} \models \neg b\}).$$

Now $\mathcal{I} \models E$ and because $\mathcal{I} \not\models \psi$, $\psi \notin E$.

We can show that E is an extension of $\langle D, W \rangle$.

Because there is an extension E such that $\psi \notin E$, ψ does not skeptically follow from $\langle D, W \rangle$. □

Nonmonotonic Logic Programs: Background

- ▶ **Answer set semantics**: a formalization of **negation-as-failure** in logic programming (**Prolog**)
- ▶ Other formalizations: **well-founded semantics**, **perfect-model semantics**, **inflationary semantics**, ...
- ▶ Can be viewed as a simpler variant of **default logic**.
- ▶ A better alternative to **the propositional logic** in some applications.

Nonmonotonic Logic Programs

- ▶ Rules $c \leftarrow b_1, \dots, b_m, \text{not } d_1, \dots, \text{not } d_k$
where $\{c, b_1, \dots, b_m, d_1, \dots, d_k\} \subseteq A$ for a set
 $A = \{a_1, \dots, a_n\}$ of propositions.
- ▶ Meaning similar to default logic: If
 1. we have derived b_1, \dots, b_m and
 2. cannot derive any of d_1, \dots, d_k ,then derive c .
- ▶ Rules without right-hand side: $c \leftarrow$
- ▶ Rules without left-hand side: $\leftarrow b_1, \dots, b_m, \text{not } d_1, \dots, \text{not } d_k$

Answer Sets – Formal Definition

- ▶ **Reduct** of a program P with respect to a set of atoms $\Delta \subseteq A$:

$$P^\Delta := \{c \leftarrow b_1, \dots, b_m \mid$$

$$(c \leftarrow b_1, \dots, b_m, \text{not } d_1, \dots, \text{not } d_k) \in P,$$

$$\{d_1, \dots, d_k\} \cap \Delta = \emptyset\}$$

- ▶ The **closure** $\text{dcl}(P) \subseteq A$ of a set P of rules **without not** is defined by iterative application of the rules in the obvious way.
- ▶ A set of propositions $\Delta \subseteq A$ is an **answer set of P** iff $\Delta = \text{dcl}(P^\Delta)$.

Examples

- ▶ $P_1 = \{a \leftarrow, b \leftarrow a, c \leftarrow b\}$
- ▶ $P_2 = \{a \leftarrow b, b \leftarrow a\}$
- ▶ $P_3 = \{p \leftarrow \text{not } p\}$
- ▶ $P_4 = \{p \leftarrow \text{not } q, q \leftarrow \text{not } p\}$
- ▶ $P_5 = \{p \leftarrow \text{not } q, q \leftarrow \text{not } p, \leftarrow p\}$

Complexity: existence of answer sets is NP-complete

1. **Membership in NP:** Guess $\Delta \subseteq A$ (*nondet. polytime*), compute P^Δ , compute its closure, compare to Δ (*everything det. polytime*).
2. **NP-hardness:** Reduction from 3SAT: an answer set exists iff clauses are satisfiable:

$$p \leftarrow \text{not } \hat{p}$$

$$\hat{p} \leftarrow \text{not } p$$

for every proposition p occurring in the clauses, and

$$\leftarrow \text{not } l'_1, \text{not } l'_2, \text{not } l'_3$$

for every clause $l_1 \vee l_2 \vee l_3$, where $l'_i = p$ if $l_i = p$ and $l'_i = \hat{p}$ if $l_i = \neg p$.

Programs for Reasoning with Answer Sets

- ▶ smodels (Niemelä & Simons), dlv (Eiter et al.), ...
- ▶ Schematic input:

```

p(X) :- not q(X).      anc(X,Y) :- par(X,Y).
q(X) :- not p(X).      anc(X,Y) :- par(X,Z), anc(Z,Y).
r(a).                  par(a,b). par(a,c). par(b,d).
r(b).                  female(a).
r(c).                  male(X) :- not(female(X)).
                        forefather(X,Y) :-
                            anc(X,Y), male(X).
  
```

Difference to the Propositional Logic

- ▶ The *ancestor* relation is **the transitive closure** of the *parent* relation.
- ▶ Transitive closure **cannot be** (concisely) represented in propositional/predicate logic.

$$par(X, Y) \rightarrow anc(X, Y)$$

$$par(X, Z) \wedge anc(Z, Y) \rightarrow anc(X, Y)$$

The above formulae only guarantee that *anc* is a *superset* of the transitive closure of *par*.

- ▶ For transitive closure one needs the minimality condition in some form: nonmonotonic logics, fixpoint logics, ...

Stratification

The reason for multiple answer sets is the fact that a may depend on b and simultaneously b may depend on a .

The lack of this kind of circular dependencies makes reasoning easier.

Definition

A logic program P is **stratified** if P can be partitioned to

$P = P_1 \cup \dots \cup P_n$ so that for all $i \in \{1, \dots, n\}$ and

$(c \leftarrow b_1, \dots, b_m, \text{not } d_1, \dots, \text{not } d_k) \in P_i$,

1. there is no **not** c in P_i and
2. there are no occurrences of c anywhere in $P_1 \cup \dots \cup P_{i-1}$.

Stratification

Theorem

A stratified program P has exactly one answer set. The unique answer set can be computed in polynomial time.

Example

Our earlier examples with more than one or no answer sets:

$$P_3 = \{p \leftarrow \text{not } p\}$$

$$P_4 = \{p \leftarrow \text{not } q, \quad q \leftarrow \text{not } p\}$$

Applications of Logic Programs

1. Simple forms of default reasoning (inheritance networks)
2. A solution to **the frame problem**: instead of using **frame axioms**, use defaults

$$a_{t+1} \leftarrow a_t, \text{not } \neg a_{t+1}$$

By default, truth-values of facts stay the same.

3. deductive databases (Datalog[⊃])
4. et cetera: Everything that can be done with propositional logic can also be done with propositional nonmonotonic logic programs.

Literature



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