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Principles of Knowledge Representation and Reasoning

Modal Logics

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May 2 & 6, 2008

Motivation for Studying Modal Logics

- Notions like **believing** and **knowing** require a more general semantics than e.g. propositional logic has.
- Some KR formalisms can be understood as (fragments of) a **propositional modal logic**.
- Application 1: spatial representation formalism **RCC8**
- Application 2: **description logics**
- Application 3: reasoning about time
- Application 4: reasoning about actions, strategies, etc.

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Motivation for Modal Logics

Often, we want to state something where we have an “embedded proposition”:

- John believes that *it is Sunday*.
- I know that $2^{10} = 1024$.

Reasoning with embedded propositions:

- *John believes that if it is Sunday, then shops are closed.*
- *John believes that it is Sunday.*
- This implies (assuming *belief* is closed under *modus ponens*):
John believes that shops are closed.

↪ How to formalize this?

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Propositional logic + operators \Box & \Diamond (*Box & Diamond*):

φ	\longrightarrow	\dots	<i>classical propositional formula</i>
		$\Box\varphi'$	<i>Box</i>
		$\Diamond\varphi'$	<i>Diamond</i>

\Box and \Diamond have the same operator precedence as \neg .

Some possible readings of $\Box\varphi$:

- Necessarily φ (alethic)
- Always φ (temporal)
- φ should be true (deontic)
- Agent A believes φ (doxastic)
- Agent A knows φ (epistemic)

\rightsquigarrow different semantics for different intended readings

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Truth-Functional Semantics?

- Could it be possible to define the meaning of $\Box\varphi$ **truth-functionally**, i.e. by referring to the truth value of φ only?
- An attempt to interpret *necessity* truth-functionally:
 - If φ is false, then $\Box\varphi$ should be false.
 - If φ is true, then ...
 - $\Box\varphi$ should be true $\Rightarrow \Box$ as the identity function
 - $\Box\varphi$ should be false $\Rightarrow \Box$ as the negation of φ
- **Note:** There are only 4 different unary Boolean functions $\{T, F\} \rightarrow \{T, F\}$.

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Semantics: The Idea

In classical propositional logic, formulae are interpreted over single interpretations and are evaluated to *true* or *false*.

In modal logics one considers *sets* of interpretations: **possible worlds** (physically possible, conceivable, ...).

Main idea:

- Consider a world (interpretation) w and a set of worlds W which are possible with respect to w .
- A classical formula (with no modal operators) φ is true with respect to (w, W) iff φ is true in w .
- $\Box\varphi$ is true wrt (w, W) iff φ is true in all worlds in W .
- $\Diamond\varphi$ is true wrt (w, W) iff φ is true in some world in W .
- Meanings of \Box and \Diamond are inter-related by: $\Diamond\varphi \equiv \neg\Box\neg\varphi$.

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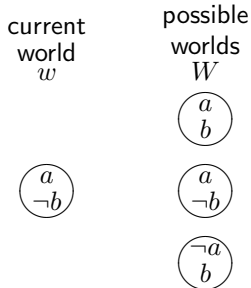
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Semantics: An Example



Examples:

- $a \wedge \neg b$ is true relative to (w, W) .
- $\Box a$ is not true relative to (w, W) .
- $\Box(a \vee b)$ is true relative to (w, W) .

Question: How to evaluate **modal** formulae in $w \in W$?

\rightsquigarrow For each world, we specify a set of possible worlds.

\rightsquigarrow frames

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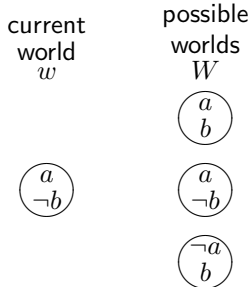
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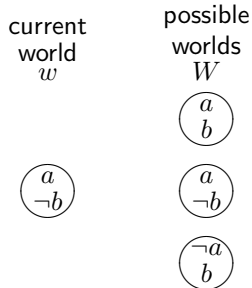
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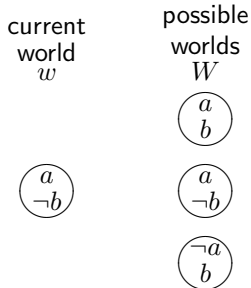
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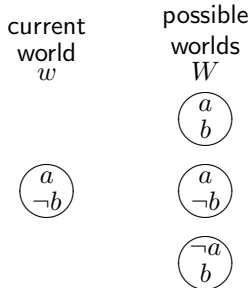
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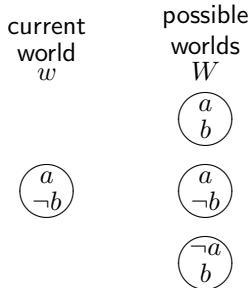
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\rightsquigarrow For each world, we specify a set of possible worlds.

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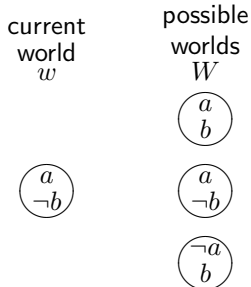
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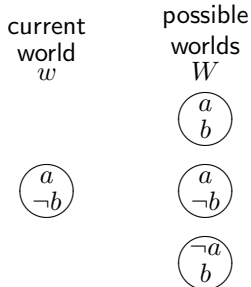
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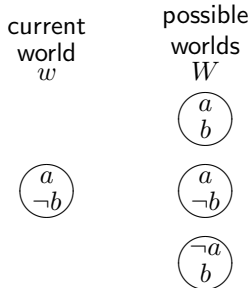
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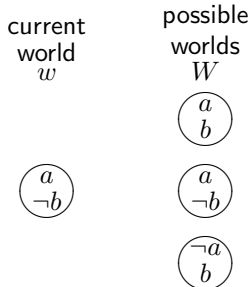
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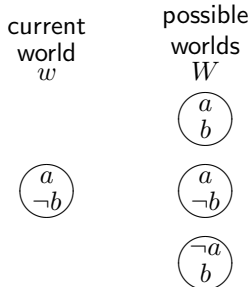
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Frames, Interpretations, and Worlds

A **(Kripke, relational) frame** is a pair $\mathcal{F} = \langle W, R \rangle$ where W is a non-empty set (of **worlds**) and $R \subseteq W \times W$ (the **accessibility relation**).

For $(w, v) \in R$ we write also $w R v$.

We say that v is an **R -successor** of w and that v is **reachable** (or R -reachable) from w .

A **(Σ)-interpretation** (or model) based on the frame $\mathcal{F} = \langle W, R \rangle$ is a triple $\mathcal{I} = \langle W, R, \pi \rangle$, where π is a function from worlds to truth assignments:

$$\pi: W \rightarrow (\Sigma \rightarrow \{T, F\}).$$

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Semantics: Truth in a World

A formula φ is **true in world w of an interpretation**
 $\mathcal{I} = \langle W, R, \pi \rangle$ under the following conditions:

$$\mathcal{I}, w \models a \quad \text{iff } \pi(w)(a) = T$$

$$\mathcal{I}, w \models \top$$

$$\mathcal{I}, w \not\models \perp$$

$$\mathcal{I}, w \models \neg\varphi \quad \text{iff } \mathcal{I}, w \not\models \varphi$$

$$\mathcal{I}, w \models \varphi \wedge \psi \quad \text{iff } \mathcal{I}, w \models \varphi \text{ and } \mathcal{I}, w \models \psi$$

$$\mathcal{I}, w \models \varphi \vee \psi \quad \text{iff } \mathcal{I}, w \models \varphi \text{ or } \mathcal{I}, w \models \psi$$

$$\mathcal{I}, w \models \varphi \rightarrow \psi \quad \text{iff if } \mathcal{I}, w \models \varphi \text{ then } \mathcal{I}, w \models \psi$$

$$\mathcal{I}, w \models \varphi \leftrightarrow \psi \quad \text{iff } \mathcal{I}, w \models \varphi \text{ if and only if } \mathcal{I}, w \models \psi$$

$$\mathcal{I}, w \models \Box\varphi \quad \text{iff } \mathcal{I}, u \models \varphi \text{ for all } u \text{ s.t. } wRu$$

$$\mathcal{I}, w \models \Diamond\varphi \quad \text{iff } \mathcal{I}, u \models \varphi \text{ for at least one } u \text{ s.t. } wRu$$

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Satisfiability and Validity

A formula φ is **satisfiable in an interpretation \mathcal{I}** (or **in a frame \mathcal{F}** , or **in a class of frames \mathcal{C}**) if there exists a world in \mathcal{I} (or an interpretation \mathcal{I} based on \mathcal{F} , or an interpretation \mathcal{I} based on a frame contained in the class \mathcal{C} , respectively) such that $\mathcal{I}, w \models \varphi$.

A formula φ is **true in an interpretation \mathcal{I}** (symbolically $\mathcal{I} \models \varphi$) if φ is true in all worlds of \mathcal{I} .

A formula φ is **valid in a frame \mathcal{F}** or **\mathcal{F} -valid** (symbolically $\mathcal{F} \models \varphi$) if φ is true in all interpretations based on \mathcal{F} .

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\mathbf{K} is the class of all frames – named after **Saul Kripke**, who invented this semantics.

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Validity: Some Examples

- 1 $\varphi \vee \neg\varphi$
- 2 $\Box(\varphi \vee \neg\varphi)$
- 3 $\Box\varphi$, if φ is a classical tautology
- 4 $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ (axiom schema K)

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Validity: Some Examples

Theorem

K is **K-valid**. $(K = \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi))$

Proof.

Let \mathcal{I} be an interpretation and let w be a world in \mathcal{I} .

Assumption: $\mathcal{I}, w \models \Box(\varphi \rightarrow \psi)$, i.e., in all worlds u with wRu , if φ is true then also ψ is. (Otherwise K is true in any case.)

If $\Box\varphi$ is false in w , then $(\Box\varphi \rightarrow \Box\psi)$ is true and K is true in w .

If $\Box\varphi$ is true in w , then both $\Box(\varphi \rightarrow \psi)$ and $\Box\varphi$ are true in w .

Hence both $\varphi \rightarrow \psi$ and φ are true in every world u accessible from w . Hence ψ is true in any such u , and therefore $w \models \Box\psi$. Since \mathcal{I} and w were arbitrary, the argument goes through for any \mathcal{I}, w , i.e., K is **K-valid**. □

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Validity: Some Examples

Theorem

K is **K-valid**. $(K = \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi))$

Proof.

Let \mathcal{I} be an interpretation and let w be a world in \mathcal{I} .

Assumption: $\mathcal{I}, w \models \Box(\varphi \rightarrow \psi)$, i.e., in all worlds u with wRu , if φ is true then also ψ is. (Otherwise K is true in any case.)

If $\Box\varphi$ is false in w , then $(\Box\varphi \rightarrow \Box\psi)$ is true and K is true in w .

If $\Box\varphi$ is true in w , then both $\Box(\varphi \rightarrow \psi)$ and $\Box\varphi$ are true in w .

Hence both $\varphi \rightarrow \psi$ and φ are true in every world u accessible from w . Hence ψ is true in any such u , and therefore $w \models \Box\psi$. Since \mathcal{I} and w were arbitrary, the argument goes through for any \mathcal{I}, w , i.e., K is **K-valid**. □

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Non-validity: Example

Proposition

$\diamond T$ is not **K**-valid.

Proof.

A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto T)\} \rangle.$$

We have $\mathcal{I}, w \not\models \diamond T$ because there is no u such that wRu . □

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Non-validity: Example

Proposition

$\Box\varphi \rightarrow \varphi$ is not **K**-valid.

Proof.

A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto F)\} \rangle.$$

We have $\mathcal{I}, w \models \Box a$, but $\mathcal{I}, w \not\models a$. □

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Non-validity: Another Example

Proposition

$\Box\varphi \rightarrow \Box\Box\varphi$ is not **K**-valid.

Proof.

A counterexample is the following interpretation:

$$\mathcal{I} = \langle \{u, v, w\}, \{(u, v), (v, w)\}, \pi \rangle$$

with

$$\pi(u) = \{a \mapsto T\}$$

$$\pi(v) = \{a \mapsto T\}$$

$$\pi(w) = \{a \mapsto F\}$$

This means $\mathcal{I}, u \models \Box a$, but $\mathcal{I}, u \not\models \Box\Box a$. □

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Accessibility and Axiom Schemata

Let us consider the following axiom schemata:

- T:** $\Box\varphi \rightarrow \varphi$ (knowledge axiom)
4: $\Box\varphi \rightarrow \Box\Box\varphi$ (positive introspection)
5: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ (or $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$: negative introspection)
B: $\varphi \rightarrow \Box\Diamond\varphi$
D: $\Box\varphi \rightarrow \Diamond\varphi$ (or $\Box\varphi \rightarrow \neg\Box\neg\varphi$: disbelief in the negation)

... and the following classes of frames, for which the accessibility relation is restricted as follows:

- T:** reflexive (wRw for each world w)
4: transitive (wRu and uRv implies wRv)
5: euclidian (wRu and wRv implies uRv)
B: symmetric (wRu implies uRw)
D: serial (for each w there exists v with wRv)

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Connection between Accessibility Relations and Axiom Schemata (1)

Theorem

Axiom schema T (4, 5, B, D) is \mathbf{T} -valid (4-, 5-, B-, or D-valid, respectively).

Proof.

For T and \mathbf{T} : Let \mathcal{F} be a frame from class \mathbf{T} . Let \mathcal{I} be an interpretation based on \mathcal{F} and let w be an arbitrary world in \mathcal{I} . If $\Box\varphi$ is not true in world w , then axiom T is true in w . If $\Box\varphi$ is true in w , then φ is true in all accessible worlds. Since the accessibility relation is **reflexive**, w is among the accessible worlds, i.e., φ is true in w . This means that also in this case T is true in w . This means, T is true in all worlds in all interpretations based on \mathbf{T} -frames. \square

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Connection between Accessibility Relations and Axiom Schemata (2)

Theorem

If T (4 , 5 , B , D) is valid in a frame \mathcal{F} , then \mathcal{F} is a **T-Frame** (**4-**, **5-**, **B-**, or **D-frame**, respectively).

Proof.

For T and **T**: Assume that \mathcal{F} is not a **T-frame**. We will construct an interpretation based on \mathcal{F} that falsifies T .

Because \mathcal{F} is not a **T-frame**, there is a world w such that not wRw . Construct an interpretation \mathcal{I} such that $w \not\models p$ and $v \models p$ for all v such that wRv .

Now $w \models \Box p$ and $w \not\models p$, and hence $w \not\models \Box p \rightarrow p$. □

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If T (4, 5, B, D) is valid in a frame \mathcal{F} , then \mathcal{F} is a **T-Frame** (4-, 5-, B-, or D-frame, respectively).

Proof.

For T and **T**: Assume that \mathcal{F} is not a **T-frame**. We will construct an interpretation based on \mathcal{F} that falsifies T .

Because \mathcal{F} is not a **T-frame**, there is a world w such that not wRw .

Construct an interpretation \mathcal{I} such that $w \not\models p$ and $v \models p$ for all v such that wRv .

Now $w \models \Box p$ and $w \not\models p$, and hence $w \not\models \Box p \rightarrow p$. □

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Different Modal Logics

Name	Property	Axiom schema
K	–	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
T	reflexivity	$\Box\varphi \rightarrow \varphi$
4	transitivity	$\Box\varphi \rightarrow \Box\Box\varphi$
5	euclidicity	$\Diamond\varphi \rightarrow \Box\Diamond\varphi$
B	symmetry	$\varphi \rightarrow \Box\Diamond\varphi$
D	seriality	$\Box\varphi \rightarrow \Diamond\varphi$

Some basic modal logics:

$$\begin{aligned} &K \\ KT4 &= S4 \\ KT5 &= S5 \\ &\vdots \end{aligned}$$

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Different Modal Logics

logics	\Box	$\Diamond = \neg\Box\neg$	K	T	4	5	B	D
alethic	necessarily	possibly	Y	Y	Y	Y	Y	Y
epistemic	known	possible	Y	Y	Y	Y	Y	Y
doxastic	believed	possible	Y	N	Y	Y	N	Y
deontic	obligatory	permitted	Y	N	Y?	Y?	N	Y
temporal	always in the future	sometimes	Y	Y/N	Y	N	N	N/Y

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- How can we show that a formula is \mathcal{C} -valid?
 - In order to show that a formula is **not \mathcal{C} -valid**, one can construct a counterexample (= an interpretation that falsifies it).
 - When trying out all ways of generating a counterexample without success, this counts as a proof of validity.
- ↪ Method of **(analytic/semantic) tableaux**

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Proof Methods

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Tableau Method

A **tableau** is a tree with nodes marked as follows:

- $w \models \varphi$,
- $w \not\models \varphi$, and
- wRv .

A branch that contains nodes marked with $w \models \varphi$ and $w \not\models \varphi$ is **closed**. All other branches are **open**. If all branches are closed, the tableau is called **closed**.

A tableau is constructed by using the **tableau rules**.

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Tableau Rules for the Propositional Logic

$$\frac{w \models \varphi \vee \psi}{w \models \varphi \mid w \models \psi}$$

$$\frac{w \not\models \varphi \vee \psi}{w \not\models \varphi \\ w \not\models \psi}$$

$$\frac{w \models \neg \varphi}{w \not\models \varphi}$$

$$\frac{w \models \varphi \wedge \psi}{w \models \varphi \\ w \models \psi}$$

$$\frac{w \not\models \varphi \wedge \psi}{w \not\models \varphi \mid w \not\models \psi}$$

$$\frac{w \not\models \neg \varphi}{w \models \varphi}$$

$$\frac{w \models \varphi \rightarrow \psi}{w \not\models \varphi \mid w \models \psi}$$

$$\frac{w \not\models \varphi \rightarrow \psi}{w \models \varphi \\ w \not\models \psi}$$

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Additional Tableau Rules for the Modal Logic **K**

$$\frac{w \models \Box\varphi}{v \models \varphi} \quad \text{if } wRv \text{ is on the branch already}$$

$$\frac{w \not\models \Box\varphi}{wRv} \quad \text{for new } v$$
$$v \not\models \varphi$$

$$\frac{w \models \Diamond\varphi}{wRv} \quad \text{for new } v$$
$$v \models \varphi$$

$$\frac{w \not\models \Diamond\varphi}{v \not\models \varphi} \quad \text{if } wRv \text{ is on the branch already}$$

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Properties of \mathbf{K} Tableaux

Proposition

If a K -tableau is closed, the truth condition at the root cannot be satisfied.

Theorem (Soundness)

If a K -tableau with root $w \not\models \varphi$ is closed, then φ is \mathbf{K} -valid.

Theorem (Completeness)

If φ is \mathbf{K} -valid, then there is a closed tableau with root $w \not\models \varphi$.

Proposition (Termination)

There are strategies for constructing \mathbf{K} -tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.

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Tableau Rules for Other Modal Logics

Proofs within more restricted classes of frames allow the use of further tableau rules.

- For reflexive (**T**) frames we may extend any branch with wRw .
- For transitive (**4**) frames we have the following additional rule:
 - If wRv and vRu are in a branch, wRu may be added to the branch.
- For serial (**D**) frames we have the following rule:
 - If there is $w \models \dots$ or $w \not\models \dots$ on a branch, then add wRv for a new world v .
- Similar rules for other properties...

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Testing Logical Consequence with Tableaux

- Let Θ be a set of formulas. When does a formula φ follow from Θ : $\Theta \models_{\mathbf{X}} \varphi$?
- Test whether in all interpretations on \mathbf{X} -frames in which Θ is true, also φ is true.
- Wouldn't there be a deduction theorem we could use?
- Example: $a \models_{\mathbf{K}} \Box a$ holds, but $a \rightarrow \Box a$ is not \mathbf{K} -valid.
- There is no deduction theorem as in the propositional logic, and logical consequence cannot be directly reduced to validity!

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Tableaux and Logical Implication

For testing logical consequence, we can use the following tableau rule:

- If w is a world on a branch and $\psi \in \Theta$, then we can add $w \models \psi$ to our branch.
- Soundness is obvious.
- Completeness is non-trivial.

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Connection between propositional modal logic and FOL?

- There are similarities between the predicate logic and propositional modal logics:
 - ① \Box vs. \forall
 - ② \Diamond vs. \exists
 - ③ the possible worlds vs. the objects of the universe
 - In fact, we can show for many propositional modal logics that they can be embedded in the predicate logic.
- ⇒ Modal logics can be understood as a sublanguage of FOL.

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Embedding Modal Logics in the Predicate Logic (1)

- 1 $\tau(p, x) = p(x)$ for propositional variables p
- 2 $\tau(\neg\phi, x) = \neg\tau(\phi, x)$
- 3 $\tau(\phi \vee \psi, x) = \tau(\phi, x) \vee \tau(\psi, x)$
- 4 $\tau(\phi \wedge \psi, x) = \tau(\phi, x) \wedge \tau(\psi, x)$
- 5 $\tau(\Box\phi, x) = \forall y(R(x, y) \rightarrow \tau(\phi, y))$ for some new y
- 6 $\tau(\Diamond\phi, x) = \exists y(R(x, y) \wedge \tau(\phi, y))$ for some new y

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Embedding Modal Logics in the Predicate Logic (2)

Theorem

ϕ is *K*-valid if and only if $\forall x \tau(\phi, x)$ is valid in the predicate logic.

Theorem

ϕ is *T*-valid if and only if in the predicate logic the logical consequence $\{\forall x R(x, x)\} \models \forall x \tau(\phi, x)$ holds.

Example

$\Box p \wedge \Diamond(p \rightarrow q) \rightarrow \Diamond q$ is *K*-valid, because

$$\forall x (\forall x' (R(x, x') \rightarrow p(x')) \wedge \exists x' (R(x, x') \wedge (p(x') \rightarrow q(x')))) \\ \rightarrow \exists x' (R(x, x') \wedge q(x'))$$

is valid in the predicate logic.

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is valid in the predicate logic.

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Embedding Modal Logics in the Predicate Logic (2)

Theorem

ϕ is K-valid if and only if $\forall x \tau(\phi, x)$ is valid in the predicate logic.

Theorem

ϕ is T-valid if and only if in the predicate logic the logical consequence $\{\forall x R(x, x)\} \models \forall x \tau(\phi, x)$ holds.

Example

$\Box p \wedge \Diamond(p \rightarrow q) \rightarrow \Diamond q$ is K-valid, because

$$\forall x (\forall x' (R(x, x') \rightarrow p(x')) \wedge \exists x' (R(x, x') \wedge (p(x') \rightarrow q(x')))) \\ \rightarrow \exists x' (R(x, x') \wedge q(x'))$$

is valid in the predicate logic.

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We only looked at some basic propositional modal logics.

There are also:

- modal first order logics (with quantification \forall and \exists and predicates)
- multi-modal logics: more than one modality, e.g. knowledge/belief operators for several agents
- temporal and dynamic logics (modalities that refer to time or programs, respectively)

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Did we really do something new? Couldn't we have done everything in propositional modal logic in FOL already?

- Yes – but now we know much more about the (restricted) system and have decidable problems!

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