

Principles of Knowledge Representation and Reasoning

Complexity Theory

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Motivation

Reminder: Basic Notions

- Algorithms and Turing Machines
- Problems, Solutions, and Complexity
- Complexity Classes P and NP
- Upper and Lower Bounds
- Polynomial Reductions
- NP-Completeness

Beyond NP

- The Class co-NP
- The Class PSPACE
- Other Classes

Oracle TMs and the Polynomial Hierarchy

- Oracle Turing-Machines
- Turing Reduction
- Complexity Classes Based on OTMs

QBE

Motivation for Using Complexity Theory

- ▶ Complexity theory can answer questions on how easy or hard a problem is
- ▶ Gives hints on what algorithms could be appropriate, e.g.:
 - ▶ algorithms for **polynomial-time problems** are usually easy to design
 - ▶ for **NP-complete** problems, backtracking and local search work well
- ▶ Gives hints on what type of algorithm will (most probably) not work
 - ▶ for problems that are believed to be harder than NP-complete ones, simple backtracking will not work
- ▶ Gives hint on what sub-problems might be interesting

Algorithms and Turing Machines

- ▶ We use **Turing machines** as formal models of algorithms
- ▶ This is justified, because:
 - ▶ we assume that Turing machines can compute all computable functions
 - ▶ the resource requirements (in term of time and memory) of a Turing machine are only polynomially worse than other models
- ▶ The regular type of Turing machine is the **deterministic** one: **DTM** (or simply **TM**)
- ▶ Often, however, we use the notion of **nondeterministic** TMs: **NDTM**

Problems, Solutions, and Complexity

- ▶ A **problem** is a set of pairs (I, A) of strings in $\{0, 1\}^*$.
 I : Instance; A : Answer.
If $A \in \{0, 1\}$: **decision problem**
- ▶ A **decision problem** is the same as a **formal language**: namely the set of strings formed by the instances with answer 1
- ▶ An algorithm **decides** (or **solves**) a problem if it computes the right answer for all instances.
- ▶ The **complexity of an algorithm** is a function

$$T: \mathbf{N} \rightarrow \mathbf{N},$$

measuring the **number of basic steps** (or memory requirement) the algorithm needs to compute an answer depending on the **size** of the instance.

- ▶ The **complexity of a problem** is the complexity of the most efficient algorithm that solves this problem.

Complexity Classes P and NP

Problems are categorized into **complexity classes** according to the requirements of computational resources:

- ▶ The class of problems decidable on **deterministic Turing machines** in **polynomial time**: **P**
- ▶ Problems in P are assumed to be **efficiently solvable** (although this might not be true if the exponent is very large)
- ▶ In practice, this notion appears to be more often reasonable than not
- ▶ The class of problems decidable on **non-deterministic Turing machines** in **polynomial time**: **NP**
- ▶ More classes are definable using other resource bounds on time and memory

Upper and Lower Bounds

- ▶ **Upper bounds** (**membership** in a class) are usually easy to prove:
 - ▶ provide an **algorithm**
 - ▶ show that the resource bounds are respected
- ▶ **Lower bounds** (**hardness** for a class) are usually difficult to show:
 - ▶ the technical tool here is the **polynomial reduction** (or any other appropriate reduction)
 - ▶ show that some hard problem can be reduced to the problem at hand

Polynomial Reductions

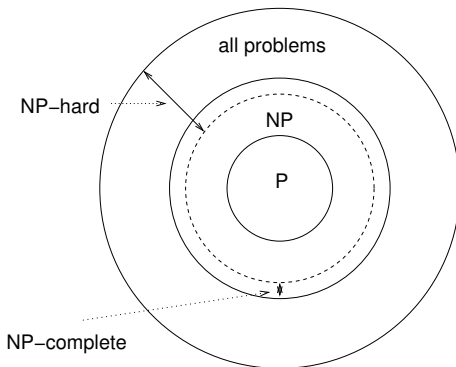
- ▶ Given two languages L_1 and L_2 , L_1 can be **polynomially reduced to** L_2 , written $L_1 \leq_p L_2$, iff there exists a polynomially computable function f such that

$$x \in L_1 \text{ iff } f(x) \in L_2$$

- ▶ It cannot be harder to decide L_1 than L_2
- ▶ L is **hard** for a class C (**C -hard**) iff all languages of this class can be reduced to L .
- ▶ L is **complete** for C (**C -complete**) iff L is C -hard and $L \in C$.

NP-complete Problems

- ▶ A problem is **NP-complete** iff it is **NP-hard** and **in NP**.
- ▶ Example: **SAT** – the satisfiability problem for propositional logic – is NP-complete (Cook/Karp)
- ▶ Membership is obvious, hardness follows because computations on a NDTM correspond to satisfying truth-assignments of certain formulae



The Complexity Class co-NP

- ▶ Note that there is some **asymmetry** in the definition of NP:
 - ▶ It is clear that we can decide **SAT** by using a **NDTM** with polynomially bounded computation
 - ▶ There exists an accepting computation of polynomial length iff the formula is satisfiable
 - ▶ What if we want to solve UNSAT, the complementary problem?
 - ▶ It seems necessary to check **all** possible truth-assignments!
- ▶ Define **co-C** = $\{L \mid \Sigma^* - L \in C\}$, provided Σ is our alphabet
- ▶ **co-NP** = $\{L \mid \Sigma^* - L \in \text{NP}\}$
- ▶ For example UNSAT, TAUT \in co-NP!
- ▶ **Note:** P is closed under complement, i.e.,

$$P \subseteq \text{NP} \cap \text{co-NP}$$

PSPACE

There are problems even more difficult than NP and co-NP.

Definition ((N)PSPACE)

PSPACE (**NPSPACE**) is the class of decision problems that can be decided on deterministic (non-deterministic) Turing machines using only **polynomially many tape cells**.

Some facts about PSPACE:

- ▶ PSPACE is **closed under complements** (as all other deterministic classes)
- ▶ PSPACE is **identical** to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- ▶ $NP \subseteq PSPACE$ (because in polynomial time one can “visit” only polynomial space, i.e., $NP \subseteq NPSPACE$)
- ▶ It is **unknown** whether $NP \neq PSPACE$, but it is **believed** that this is true.

PSPACE-completeness

Definition (PSPACE-completeness)

A decision problem (or language) is **PSPACE-complete**, if it is in PSPACE and all other problems in PSPACE can be polynomially reduced to it.

Intuitively, **PSPACE-complete** problems are the “hardest” problems in PSPACE (similar to NP-completeness). They appear to be “harder” than **NP-complete** problems from a *practical point of view*.

An example for a PSPACE-complete problem is the **NFA equivalence problem**:

Instance: *Two non-deterministic finite state automata A_1 and A_2 .*

Question: *Are the languages accepted by A_1 and A_2 identical?*

Other Complexity Classes . . .

- ▶ There are complexity classes **above PSPACE** (EXPTIME, EXPSPACE, NEXPTIME, DEXPTIME . . .)
- ▶ there are (infinitely many) classes **between NP and PSPACE** (the **polynomial hierarchy** defined by **oracle machines**)
- ▶ there are (infinitely many) classes **inside P** (circuit classes with different depths)
- ▶ and for most of the classes **we do not know** whether the containment relationships are **strict**

Oracle Turing Machines

- ▶ An **Oracle Turing machine** ((N)OTM) is a Turing machine (DTM, NDTM) with the possibility to query an **oracle** (i. e., a different Turing machine **without resource restrictions**) whether it accepts or rejects a given string.
- ▶ **Computation by the oracle does not cost anything!**
- ▶ **Formalization:**
 - ▶ a tape onto which strings for the oracle are written,
 - ▶ a yes/no answer from the oracle depending on whether it accepts or rejects the input string.
- ▶ Usage of OTMs answers **what-if questions**: What if we could solve the oracle-problem efficiently?

Turing Reductions

- ▶ OTMs allow us to define a more general type of reduction
- ▶ Idea: The “classical” reduction can be seen as calling a subroutine once.
- ▶ L_1 is Turing-reducible to L_2 , symbolically $L_1 \leq_T L_2$, if there exists a poly-time OTM that decides L_1 by using an oracle for L_2 .
- ▶ Polynomial reducibility implies Turing reducibility, but not *vice versa*!
- ▶ NP-hardness and co-NP-hardness with respect to Turing reducibility are equivalent!
- ▶ Turing reducibility can also be applied to general search problems!

Complexity Classes Based on Oracle TMs

1. P^{NP} = decision problems solved by poly-time DTMs with an oracle for a decision problem in NP.
2. NP^{NP} = decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
3. $co-NP^{NP}$ = complements of decision problems solved by poly-time NDTMs with an oracle for a decision problem in NP.
4. $NP^{NP^{NP}}$ = ...

... and so on

Example

- ▶ Consider the **Minimum Equivalent Expression (MEE)** problem:

Instance: *A well-formed Boolean formula ϕ using the standard connectives (not \leftrightarrow) and a nonnegative integer K .*

Question: *Is there a well-formed Boolean formula ϕ' that contains K or fewer literal occurrences and that is logical equivalent to ϕ ?*

- ▶ This problem is NP-hard (wrt. to Turing reductions).
- ▶ It does not appear to be NP-complete
- ▶ We could guess a formula and then use a SAT-oracle
- ▶ $\text{MEE} \in \text{NP}^{\text{NP}}$.

The Polynomial Hierarchy

The complexity classes based on OTMs form an infinite hierarchy.

The polynomial hierarchy PH

$$\begin{array}{lll} \Sigma_0^P = P & \Pi_0^P = P & \Delta_0^P = P \\ \Sigma_{i+1}^P = \text{NP}^{\Sigma_i^P} & \Pi_{i+1}^P = \text{co-}\Sigma_{i+1}^P & \Delta_{i+1}^P = P^{\Sigma_i^P} \end{array}$$

- ▶ $\text{PH} = \bigcup_{i \geq 0} (\Sigma_i^P \cup \Pi_i^P \cup \Delta_i^P) \subseteq \text{PSPACE}$
- ▶ $\text{NP} = \Sigma_1^P$
- ▶ $\text{co-NP} = \Pi_1^P$

Quantified Boolean Formulae: Definition

- ▶ If ϕ is a propositional formula, P is the set of Boolean variables used in ϕ and σ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma\phi$ is a **QBF**.
- ▶ A formula $\exists x\phi$ is **true** if and only if $\phi[\top/x] \vee \phi[\perp/x]$ is true. (Equivalently, $\phi[\top/x]$ is true **or** $\phi[\perp/x]$ is true.)
- ▶ A formula $\forall x\phi$ is **true** if and only if $\phi[\top/x] \wedge \phi[\perp/x]$ is true. (Equivalently, $\phi[\top/x]$ is true **and** $\phi[\perp/x]$ is true.)
- ▶ This definition directly leads to an AND/OR tree traversal algorithm for evaluating QBF.

Quantified Boolean Formulae: Definition

The **evaluation problem of QBF** generalizes both the *satisfiability* and *validity/tautology problems* of propositional logic.

The latter are respectively **NP-complete** and **co-NP-complete** whereas the former is **PSPACE-complete**.

Example

The formulae $\forall x \exists y (x \leftrightarrow y)$ and $\exists x \exists y (x \wedge y)$ are true.

Example

The formulae $\exists x \forall y (x \leftrightarrow y)$ and $\forall x \forall y (x \vee y)$ are false.

The Polynomial Hierarchy: Connection to QBF

Truth of QBFs with prefix $\overbrace{\forall \exists \forall \dots}^i$ is Π_i^P -complete.

Truth of QBFs with prefix $\overbrace{\exists \forall \exists \dots}^i$ is Σ_i^P -complete.

Special cases corresponding to **SAT** and **TAUT**:

The truth of QBFs with prefix $\exists x_1^1 \dots x_n^1$ is $\text{NP} = \Sigma_1^P$ -complete.

The truth of QBFs with prefix $\forall x_1^1 \dots x_n^1$ is $\text{co-NP} = \Pi_1^P$ -complete.

Literature



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