

# Principles of Knowledge Representation and Reasoning

## Predicate Logic

Bernhard Nebel, Malte Helmert and Stefan Wöfl

Albert-Ludwigs-Universität Freiburg

April 25, 2008

# Principles of Knowledge Representation and Reasoning

April 25, 2008 — Predicate Logic

Motivation

Syntax

Semantics

- Interpretations

- Variable Maps

- Definition of Truth

- Terminology

- Free and Bound Variables

- Open and Closed Formulae

Literature

# Why First-Order Logic (FOL)?

- ▶ In propositional logic, the only building blocks are atomic propositions.
- ▶ We cannot talk about the internal structures of these propositions.
- ▶ **Example:**
  - ▶ *All CS students know formal logic*
  - ▶ *Peter is a CS student*
  - ▶ Therefore, *Peter knows formal logic*
  - ▶ Not possible in propositional logic
- ▶ **Idea:** We introduce **predicates**, **functions**, **object variables** and **quantifiers**.

# Syntax

- ▶ **variable** symbols:  $x, y, z, \dots$
- ▶  $n$ -ary **function** symbols:  $f, g, \dots$
- ▶ **constant** symbols:  $a, b, c, \dots$
- ▶  $n$ -ary **predicate** symbols:  $P, Q, \dots$
- ▶ logical symbols:  $\forall, \exists, =, \neg, \wedge, \dots$

<b>Terms</b>	$t$	$\longrightarrow$	$x$	variable
			$f(t_1, \dots, t_n)$	function application
			$a$	constant

<b>Formulae</b>	$\varphi$	$\longrightarrow$	$P(t_1, \dots, t_n)$	atomic formulae
			$t = t'$	identity formulae
			$\dots$	propositional connectives
			$\forall x \varphi'$	universal quantification
			$\exists x \varphi'$	existential quantification

**ground** term, etc.: term, etc. without variable occurrences

## Semantics: Idea

- ▶ In FOL, the **universe of discourse** consists of objects, functions over these objects, and relations over these objects.
- ▶ Function symbols are mapped to functions, predicate symbols are mapped to relations, and terms to objects.
- ▶ **Notation**: Instead of  $\mathcal{I}(x)$  we write  $x^{\mathcal{I}}$ .
- ▶ **Note**: Usually one considers **all possible** non-empty universes. (However, sometimes the interpretations are restricted to particular domains, e.g. integers or real numbers.)
- ▶ Satisfiability and validity is then considered wrt all these universes.

## Formal Semantics: Interpretations

**Interpretations:**  $\mathcal{I} = \langle \mathcal{D}, \cdot^{\mathcal{I}} \rangle$  with  $\mathcal{D}$  being an arbitrary non-empty set and  $\cdot^{\mathcal{I}}$  being a function which maps

- ▶  $n$ -ary function symbols  $f$  to  $n$ -ary functions  $f^{\mathcal{I}} \in [\mathcal{D}^n \rightarrow \mathcal{D}]$ ,
- ▶ constant symbols  $a$  to objects  $a^{\mathcal{I}} \in \mathcal{D}$ , and
- ▶  $n$ -ary predicates  $P$  to  $n$ -ary relations  $P^{\mathcal{I}} \subseteq \mathcal{D}^n$ .

**Interpretation** of ground terms:

$$(f(t_1, \dots, t_n))^{\mathcal{I}} = f^{\mathcal{I}}(t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}}) (\in \mathcal{D})$$

**Truth** of ground atoms:

$$\mathcal{I} \models P(t_1, \dots, t_n) \quad \text{iff} \quad \langle t_1^{\mathcal{I}}, \dots, t_n^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

## Examples

$$\mathcal{D} = \{d_1, \dots, d_n\}, n \geq 2$$

$$a^{\mathcal{I}} = d_1$$

$$b^{\mathcal{I}} = d_2$$

$$\text{eye}^{\mathcal{I}} = \{d_1\}$$

$$\text{red}^{\mathcal{I}} = \mathcal{D}$$

$$\mathcal{I} \models \text{red}(b)$$

$$\mathcal{I} \not\models \text{eye}(b)$$

$$\mathcal{D} = \{1, 2, 3, \dots\}$$

$$1^{\mathcal{I}} = 1$$

$$2^{\mathcal{I}} = 2$$

$$\vdots$$

$$\text{even}^{\mathcal{I}} = \{2, 4, 6, \dots\}$$

$$\text{succ}^{\mathcal{I}} = \{(1 \mapsto 2), (2 \mapsto 3), \dots\}$$

$$\mathcal{I} \not\models \text{even}(3)$$

$$\mathcal{I} \models \text{even}(\text{succ}(3))$$

# Formal Semantics: Variable Maps

$V$  is the set of variables. Function  $\alpha: V \rightarrow \mathcal{D}$  is a **variable map**.

**Notation:**  $\alpha[x/d]$  is identical to  $\alpha$  except for  $x$  where  $\alpha[x/d](x) = d$ .

Interpretation of terms under  $\mathcal{I}, \alpha$ :

$$\begin{aligned} x^{\mathcal{I}, \alpha} &= \alpha(x) \\ a^{\mathcal{I}, \alpha} &= a^{\mathcal{I}} \\ (f(t_1, \dots, t_n))^{\mathcal{I}, \alpha} &= f^{\mathcal{I}}(t_1^{\mathcal{I}, \alpha}, \dots, t_n^{\mathcal{I}, \alpha}) \end{aligned}$$

Truth of atomic formulae:

$$\mathcal{I}, \alpha \models P(t_1, \dots, t_n) \quad \text{iff} \quad \langle t_1^{\mathcal{I}, \alpha}, \dots, t_n^{\mathcal{I}, \alpha} \rangle \in P^{\mathcal{I}}$$

**Example** (cont'd):

$$\alpha = \{x \mapsto d_1, y \mapsto d_2\} \quad \mathcal{I}, \alpha \models \text{red}(x) \quad \mathcal{I}, \alpha[y/d_1] \models \text{eye}(y)$$

# Formal Semantics: Truth

Truth of  $\varphi$  by  $\mathcal{I}$  under  $\alpha$  ( $\mathcal{I}, \alpha \models \varphi$ ) is defined as follows.

$\mathcal{I}, \alpha \models P(t_1, \dots, t_n)$	iff $\langle t_1^{\mathcal{I}, \alpha}, \dots, t_n^{\mathcal{I}, \alpha} \rangle \in P^{\mathcal{I}}$
$\mathcal{I}, \alpha \models t_1 = t_2$	iff $t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha}$
$\mathcal{I}, \alpha \models \neg \varphi$	iff $\mathcal{I}, \alpha \not\models \varphi$
$\mathcal{I}, \alpha \models \varphi \wedge \psi$	iff $\mathcal{I}, \alpha \models \varphi$ and $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \varphi \vee \psi$	iff $\mathcal{I}, \alpha \models \varphi$ or $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \varphi \rightarrow \psi$	iff if $\mathcal{I}, \alpha \models \varphi$ , then $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \varphi \leftrightarrow \psi$	iff $\mathcal{I}, \alpha \models \varphi$ , iff $\mathcal{I}, \alpha \models \psi$
$\mathcal{I}, \alpha \models \forall x \varphi$	iff $\mathcal{I}, \alpha[x/d] \models \varphi$ for all $d \in \mathcal{D}$
$\mathcal{I}, \alpha \models \exists x \varphi$	iff $\mathcal{I}, \alpha[x/d] \models \varphi$ for some $d \in \mathcal{D}$

# Examples

$$\Theta = \left\{ \begin{array}{l} \text{eye}(a), \text{eye}(b) \\ \forall x(\text{eye}(x) \rightarrow \text{red}(x)) \end{array} \right\}$$

$$\mathcal{D} = \{d_1, \dots, d_n\}, \quad n > 1$$

$$a^{\mathcal{I}} = d_1$$

$$b^{\mathcal{I}} = d_1$$

$$\text{eye}^{\mathcal{I}} = \{d_1\}$$

$$\text{red}^{\mathcal{I}} = \mathcal{D}$$

$$\alpha = \{(x \mapsto d_1), (y \mapsto d_2)\}$$

## Questions:

$\mathcal{I}, \alpha \models \text{eye}(b) \vee \neg \text{eye}(b)$ ? **Yes**

$\mathcal{I}, \alpha \models \text{eye}(x) \rightarrow$   
 $\text{eye}(x) \vee \text{eye}(y)$ ? **Yes**

$\mathcal{I}, \alpha \models \text{eye}(x) \rightarrow \text{eye}(y)$ ? **No**

$\mathcal{I}, \alpha \models \text{eye}(a) \wedge \text{eye}(b)$ ? **Yes**

$\mathcal{I}, \alpha \models \forall x(\text{eye}(x) \rightarrow \text{red}(x))$ ?  
**Yes**

$\mathcal{I}, \alpha \models \Theta$ ? **Yes**

# Terminology

$\mathcal{I}, \alpha$  is a **model** of  $\varphi$  iff

$$\mathcal{I}, \alpha \models \varphi.$$

A formula can be **satisfiable**, **unsatisfiable**, **falsifiable**, **valid**, ...

Two formulae  $\varphi$  and  $\psi$  are **logically equivalent** (symb.:  $\varphi \equiv \psi$ ) iff for all  $\mathcal{I}, \alpha$ :

$$\mathcal{I}, \alpha \models \varphi \text{ iff } \mathcal{I}, \alpha \models \psi.$$

**Note:**  $P(x) \not\equiv P(y)$ !

**Logical implication** is also analogous to propositional logic:

$$\Theta \models \varphi \text{ iff for all } \mathcal{I}, \alpha \text{ s.t. } \mathcal{I}, \alpha \models \Theta \text{ also } \mathcal{I}, \alpha \models \varphi.$$

# Free and Bound Variables

Variables can be **free** or **bound** (by a quantifier) in a formula:

$$\text{free}(x) = \{x\}$$

$$\text{free}(f(t_1, \dots, t_n)) = \text{free}(t_1) \cup \dots \cup \text{free}(t_n)$$

$$\text{free}(t_1 = t_2) = \text{free}(t_1) \cup \text{free}(t_2)$$

$$\text{free}(P(t_1, \dots, t_n)) = \text{free}(t_1) \cup \dots \cup \text{free}(t_n)$$

$$\text{free}(\neg\varphi) = \text{free}(\varphi)$$

$$\text{free}(\varphi * \psi) = \text{free}(\varphi) \cup \text{free}(\psi), \text{ for } * = \vee, \wedge, \rightarrow, \leftrightarrow$$

$$\text{free}(\Xi x \varphi) = \text{free}(\varphi) \setminus \{x\}, \text{ for } \Xi = \forall, \exists$$

**Example:**  $\forall x (\mathbf{R}(\mathbf{y}, \mathbf{z}) \wedge \exists y (\neg P(y, x) \vee \mathbf{R}(y, \mathbf{z})))$

Framed occurrences are free, all others are bound.

# Open & Closed Formulae

- ▶ Formulae without free variables are called **closed formulae** or **sentences**. Formulae with free variables are called **open formulae**.
- ▶ Closed formulae are all we need when we want to state something about the world. Open formulae (and variable maps) are only necessary for technical reasons (semantics of  $\forall$  and  $\exists$ ).
- ▶ Note that *logical equivalence*, *satisfiability*, and *entailment* are independent from variable maps if we consider only closed formulae.
- ▶ For closed formulae, we omit  $\alpha$  in connection with  $\models$ :

$$\mathcal{I} \models \varphi.$$

# Important Theorems

## Theorem (Compactness)

Let  $\Phi \cup \{\psi\}$  be a set of closed formulae.

- (a)  $\Phi \models \psi$  iff there exists a finite subset  $\Phi' \subseteq \Phi$  s. t.  $\Phi' \models \psi$ .
- (b)  $\Phi$  is satisfiable iff each finite subset  $\Phi' \subseteq \Phi$  is satisfiable.

## Theorem (Löwenheim-Skolem)

Each countable set of closed formulae that is satisfiable is satisfiable on a countable domain.

# Literature



Harry R. Lewis and Christos H. Papadimitriou.

*Elements of the Theory of Computation.*

Prentice-Hall, Englewood Cliffs, NJ, 1981 (Chapters 8 & 9).



Volker Sperschneider and Grigorios Antoniou.

*Logic – A Foundation for Computer Science.*

Addison-Wesley, Reading, MA, 1991 (Chapters 1–3).



H.-P. Ebbinghaus, J. Flum, and W. Thomas.

*Einführung in die mathematische Logik.*

Wissenschaftliche Buchgesellschaft, Darmstadt, 1986.



U. Schöning.

*Logik für Informatiker.*

Spektrum-Verlag.