

# Principles of Knowledge Representation and Reasoning

## Propositional Logic

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Why Logic?

## Why Logic?

- ▶ Logic is one of the best developed systems for **representing knowledge**.
- ▶ Can be used for analysis, design and specification.
- ▶ Understanding formal logic is a prerequisite for understanding most research papers in KRR.

Why Logic?

## The Right Logic. . .

- ▶ Logics of different **orders** (1st, 2nd, ...)
- ▶ **Modal** logics
  - ▶ epistemic
  - ▶ temporal
  - ▶ dynamic (program)
  - ▶ multi-modal logics
  - ▶ ...
- ▶ **Many-valued** logics
- ▶ **Conditional** logics
- ▶ **Nonmonotonic** logics
- ▶ **Linear** logics
- ▶ ...

## The Logical Approach

- ▶ Define a **formal language**: logical & non-logical symbols, syntax rules
- ▶ Provide language with **compositional semantics**
  - ▶ Fix **universe** of discourse
  - ▶ Specify how the non-logical symbols can be **interpreted**: **interpretation**
  - ▶ Rules how to **combine** interpretation of single symbols
  - ▶ **Satisfying interpretation** = **model**
  - ▶ Semantics often entails concept of **logical implication/entailment**
- ▶ Specify a **calculus** that allows to **derive** new formulae from old ones – according to the entailment relation

## Propositional Logic: Main Ideas

- ▶ **Non-logical symbols**: propositional **variables** or **atoms**
  - ▶ representing **propositions** which cannot be decomposed
  - ▶ which can be **true** or **false** (for example: “Snow is white”, “It rains”)
- ▶ **Logical symbols**: propositional connectives such as: **and** ( $\wedge$ ), **or** ( $\vee$ ), and **not** ( $\neg$ )
- ▶ **Formulae**: built out of atoms and connectives
- ▶ **Universe of discourse**: truth values

## Syntax

Countable alphabet  $\Sigma$  of **atomic propositions**:  $a, b, c, \dots$

**Propositional formulae** are built according to the following **rule**:

$\varphi$	$\rightarrow$	$a$	<i>atomic formula</i>
		$\perp$	<i>falsity</i>
		$\top$	<i>truth</i>
		$(\neg\varphi')$	<i>negation</i>
		$(\varphi' \wedge \varphi'')$	<i>conjunction</i>
		$(\varphi' \vee \varphi'')$	<i>disjunction</i>
		$(\varphi' \rightarrow \varphi'')$	<i>implication</i>
		$(\varphi' \leftrightarrow \varphi'')$	<i>equivalence</i>

Parentheses can be omitted if no ambiguity arises.

**Operator precedence**:  $\neg > \wedge > \vee > \rightarrow = \leftrightarrow$ .

## Semantics: Idea

- ▶ Atomic propositions can be **true** ( $1, T$ ) or **false** ( $0, F$ ).
- ▶ Provided the truth values of the atoms have been fixed (**truth assignment** or **interpretation**), the truth value of a formula can be computed from the truth values of the atoms and the connectives.
- ▶ **Example**:

$$(a \vee b) \wedge c$$

is true *iff*  $c$  is true and, additionally,  $a$  or  $b$  is true.

Logical implication can then be defined as follows:

- ▶  $\varphi$  is **implied** by a set of formulae  $\Theta$  iff  $\varphi$  is true for all truth assignments (world states) that make all formulae in  $\Theta$  true.

## Formal Semantics

An **interpretation** or **truth assignment** over  $\Sigma$  is a function:

$$\mathcal{I}: \Sigma \rightarrow \{T, F\}.$$

A formula  $\psi$  is **true under**  $\mathcal{I}$  or is **satisfied by**  $\mathcal{I}$  (symb.  $\mathcal{I} \models \psi$ ):

$$\mathcal{I} \models a \quad \text{iff} \quad \mathcal{I}(a) = T$$

$$\mathcal{I} \models \top$$

$$\mathcal{I} \not\models \perp$$

$$\mathcal{I} \models \neg\varphi \quad \text{iff} \quad \mathcal{I} \not\models \varphi$$

$$\mathcal{I} \models \varphi \wedge \varphi' \quad \text{iff} \quad \mathcal{I} \models \varphi \text{ and } \mathcal{I} \models \varphi'$$

$$\mathcal{I} \models \varphi \vee \varphi' \quad \text{iff} \quad \mathcal{I} \models \varphi \text{ or } \mathcal{I} \models \varphi'$$

$$\mathcal{I} \models \varphi \rightarrow \varphi' \quad \text{iff} \quad \text{if } \mathcal{I} \models \varphi, \text{ then } \mathcal{I} \models \varphi'$$

$$\mathcal{I} \models \varphi \leftrightarrow \varphi' \quad \text{iff} \quad \mathcal{I} \models \varphi \text{ if and only if } \mathcal{I} \models \varphi'$$

## Example

Given

$$\mathcal{I}: a \mapsto T, b \mapsto F, c \mapsto F, d \mapsto T,$$

Is  $((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge c) \vee (c \wedge \neg d))$  true or false?

$$((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge c) \vee (c \wedge \neg d))$$

$$((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge c) \vee (c \wedge \neg d))$$

$$((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge c) \vee (c \wedge \neg d))$$

$$((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge c) \vee (c \wedge \neg d))$$

$$((a \vee b) \leftrightarrow (c \vee d)) \wedge (\neg(a \wedge c) \vee (c \wedge \neg d))$$

## Terminology

An interpretation  $\mathcal{I}$  is a **model** of  $\varphi$  iff

$$\mathcal{I} \models \varphi$$

A formula  $\varphi$  is

- ▶ **satisfiable** if there is an  $\mathcal{I}$  such that  $\mathcal{I} \models \varphi$ ;
- ▶ **unsatisfiable**, otherwise; and
- ▶ **valid** if  $\mathcal{I} \models \varphi$  for each  $\mathcal{I}$ ;
- ▶ **falsifiable**, otherwise.

Two formulae  $\varphi$  and  $\psi$  are **logically equivalent** (symb.  $\varphi \equiv \psi$ ) if for all interpretations  $\mathcal{I}$ ,

$$\mathcal{I} \models \varphi \text{ iff } \mathcal{I} \models \psi.$$

## Examples

Satisfiable, unsatisfiable, falsifiable, valid?

$$(a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee d) \wedge (\neg a \vee b \vee \neg d)$$

$$\rightsquigarrow \text{ satisfiable: } a \mapsto T, b \mapsto F, d \mapsto F, \dots$$

$$\rightsquigarrow \text{ falsifiable: } a \mapsto F, b \mapsto F, c \mapsto T, \dots$$

$$((\neg a \rightarrow \neg b) \rightarrow (b \rightarrow a))$$

$$\rightsquigarrow \text{ satisfiable: } a \mapsto T, b \mapsto T$$

$$\rightsquigarrow \text{ valid: Consider all interpretations or argue about falsifying ones.}$$

Equivalence?  $\neg(a \vee b) \equiv \neg a \wedge \neg b$

$$\rightsquigarrow \text{ Of course, equivalent (de Morgan).}$$

## Some Obvious Consequences

### Proposition

$\varphi$  is valid iff  $\neg\varphi$  is unsatisfiable and  $\varphi$  is satisfiable iff  $\neg\varphi$  is falsifiable.

### Proposition

$\varphi \equiv \psi$  iff  $\varphi \leftrightarrow \psi$  is valid.

### Theorem

If  $\varphi \equiv \psi$  and  $\chi'$  results from substituting  $\varphi$  by  $\psi$  in  $\chi$ , then  $\chi' \equiv \chi$ .

## Some Equivalences

simplifications	$\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$	$\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
idempotency	$\varphi \vee \varphi \equiv \varphi$	$\varphi \wedge \varphi \equiv \varphi$
commutativity	$\varphi \vee \psi \equiv \psi \vee \varphi$	$\varphi \wedge \psi \equiv \psi \wedge \varphi$
associativity	$(\varphi \vee \psi) \vee \chi \equiv \varphi \vee (\psi \vee \chi)$	$(\varphi \wedge \psi) \wedge \chi \equiv \varphi \wedge (\psi \wedge \chi)$
absorption	$\varphi \vee (\varphi \wedge \psi) \equiv \varphi$	$\varphi \wedge (\varphi \vee \psi) \equiv \varphi$
distributivity	$\varphi \wedge (\psi \vee \chi) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$	$\varphi \vee (\psi \wedge \chi) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \chi)$
double negation	$\neg\neg\varphi \equiv \varphi$	
constants	$\neg\top \equiv \perp$	$\neg\perp \equiv \top$
De Morgan	$\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$	$\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$
truth	$\varphi \vee \top \equiv \top$	$\varphi \wedge \top \equiv \varphi$
falsity	$\varphi \vee \perp \equiv \varphi$	$\varphi \wedge \perp \equiv \perp$
taut./contrad.	$\varphi \vee \neg\varphi \equiv \top$	$\varphi \wedge \neg\varphi \equiv \perp$

## How Many Different Formulae Are There ...

... for a given finite alphabet  $\Sigma$ ?

- ▶ Infinitely many:  $a, a \vee a, a \wedge a, a \vee a \vee a, \dots$
- ▶ How many different logically distinguishable (not equivalent) formulae?
  - ▶ For  $\Sigma$  with  $n = |\Sigma|$ , there are  $2^n$  different interpretations.
  - ▶ A formula can be characterized by its set of models (if two formulae are not logically equivalent, then their sets of models differ).
  - ▶ There are  $2^{(2^n)}$  different sets of interpretations.
  - ▶ There are  $2^{(2^n)}$  (logical) equivalence classes of formulae.

## Logical Implication

- ▶ Extension of the relation  $\models$  to sets  $\Theta$  of formulae:

$$\mathcal{I} \models \Theta \text{ iff } \mathcal{I} \models \varphi \text{ for all } \varphi \in \Theta.$$

- ▶  $\varphi$  is **logically implied** by  $\Theta$  (symbolically  $\Theta \models \varphi$ ) iff  $\varphi$  is true in all models of  $\Theta$ :

$$\Theta \models \varphi \text{ iff } \mathcal{I} \models \varphi \text{ for all } \mathcal{I} \text{ such that } \mathcal{I} \models \Theta$$

- ▶ Some consequences:
  - ▶ **Deduction theorem:**  $\Theta \cup \{\varphi\} \models \psi$  iff  $\Theta \models \varphi \rightarrow \psi$
  - ▶ **Contraposition:**  $\Theta \cup \{\varphi\} \models \neg\psi$  iff  $\Theta \cup \{\psi\} \models \neg\varphi$
  - ▶ **Contradiction:**  $\Theta \cup \{\varphi\}$  is unsatisfiable iff  $\Theta \models \neg\varphi$

## Normal Forms

### Terminology:

- ▶ Atomic formulae  $a$ , negated atomic formulae  $\neg a$ , truth  $\top$  and falsity  $\perp$  are **literals**.
- ▶ A disjunction of literals is a **clause**.
- ▶ If  $\neg$  only occurs in front of an atom and there are no occurrences of  $\rightarrow$  and  $\leftrightarrow$ , the formula is in **negation normal form (NNF)**.  
Example:  $(\neg a \vee \neg b) \wedge c$ , but not:  $\neg(a \wedge b) \wedge c$
- ▶ A conjunction of clauses is in **conjunctive normal form (CNF)**.  
Example:  $(a \vee b) \wedge (\neg a \vee c)$
- ▶ The dual form (disjunction of conjunctions of literals) is in **disjunctive normal form (DNF)**.  
Example:  $(a \wedge b) \vee (\neg a \wedge c)$

## Negation Normal Form

### Theorem

For each propositional formula there is a logically equivalent formula in NNF.

### Proof.

First eliminate  $\rightarrow$  and  $\leftrightarrow$  by the appropriate equivalences. The rest of the proof is by structural induction.

Base case: Claim is true for  $a$ ,  $\neg a$ ,  $\top$ ,  $\perp$ .

Inductive case: Assume claim is true for all formulae  $\varphi$  (up to a certain number of connectives) and call its NNF  $\text{nnf}(\varphi)$ .

- ▶  $\text{nnf}(\varphi \wedge \psi) = \text{nnf}(\varphi) \wedge \text{nnf}(\psi)$
- ▶  $\text{nnf}(\varphi \vee \psi) = \text{nnf}(\varphi) \vee \text{nnf}(\psi)$
- ▶  $\text{nnf}(\neg(\varphi \wedge \psi)) = \text{nnf}(\neg\varphi) \vee \text{nnf}(\neg\psi)$
- ▶  $\text{nnf}(\neg(\varphi \vee \psi)) = \text{nnf}(\neg\varphi) \wedge \text{nnf}(\neg\psi)$
- ▶  $\text{nnf}(\neg(\neg\varphi)) = \text{nnf}(\varphi)$

□

## Conjunctive Normal Form

### Theorem

For each propositional formula there are logically equivalent formulae in CNF and DNF, respectively.

### Proof.

The claim is true for  $a$ ,  $\neg a$ ,  $\top$ ,  $\perp$ .

Let us assume it is true for all formulae  $\varphi$  (up to a certain number of connectives) and call its CNF  $\text{cnf}(\varphi)$  (and its DNF  $\text{dnf}(\varphi)$ ).

- ▶  $\text{cnf}(\neg\varphi) = \text{nnf}(\neg \text{dnf}(\varphi))$  and  $\text{cnf}(\varphi \wedge \psi) = \text{cnf}(\varphi) \wedge \text{cnf}(\psi)$ .
- ▶ Assume  $\text{cnf}(\varphi) = \bigwedge_i \chi_i$  and  $\text{cnf}(\psi) = \bigwedge_j \rho_j$  with  $\chi_i, \rho_j$  being clauses. Then

$$\begin{aligned} \text{cnf}(\varphi \vee \psi) &= \text{cnf}\left(\left(\bigwedge_i \chi_i\right) \vee \left(\bigwedge_j \rho_j\right)\right) \\ &= \bigwedge_i \bigwedge_j (\chi_i \vee \rho_j) \quad (\text{by distributivity}) \end{aligned}$$

□

## How to Decide Properties of Formulae

How do we decide whether a formula is satisfiable, unsatisfiable, valid, or falsifiable?

**Note:** Satisfiability and falsifiability are **NP-complete**. Validity and unsatisfiability are **co-NP-complete**.

- ▶ A CNF formula is valid iff all clauses contain two complementary literals or  $\top$ .
- ▶ A DNF formula is satisfiable iff one disjunct does not contain  $\perp$  or two complementary literals.
- ▶ However, transformation to CNF or DNF may take exponential time (and space!).
- ▶ One can try out all truth assignments.
- ▶ One can test systematically for satisfying truth assignments (backtracking search)  
 $\rightsquigarrow$  **Davis-Putnam-Logemann-Loveland procedure (DPLL)**.

## Deciding Entailment

- ▶ We want to decide  $\Theta \models \varphi$ .
- ▶ Use deduction theorem and reduce to validity:

$$\Theta \models \varphi \text{ iff } \bigwedge \Theta \rightarrow \varphi \text{ is valid.}$$

- ▶ Now negate and test for unsatisfiability using DPLL.
- ▶ Different approach: Try to **derive**  $\varphi$  from  $\Theta$  – find a **proof** of  $\varphi$  from  $\Theta$ .
- ▶ Use **inference rules** to **derive** new formulae from  $\Theta$ . Continue to deduce new formulae until  $\varphi$  can be deduced.
- ▶ One particular calculus: **resolution**.

## Resolution: Representation

- ▶ We assume that all formulae are in CNF.
  - ▶ Can be generated using the described method.
  - ▶ Often formulae are already close to CNF.
  - ▶ There is a “cheap” conversion from arbitrary formulae to CNF that **preserves satisfiability** – which is enough as we will see.
- ▶ More convenient representation:
  - ▶ CNF formula is represented as a set.
  - ▶ Each clause is a set of literals.
  - ▶  $(a \vee \neg b) \wedge (\neg a \vee c) \rightsquigarrow \{\{a, \neg b\}, \{\neg a, c\}\}$
- ▶ Empty clause (symbolically  $\square$ ) and empty set of clauses (symbolically  $\emptyset$ ) are different!

## Resolution: The Inference Rule

Let  $l$  be a literal and  $\bar{l}$  its complement.

The resolution rule

$$\frac{C_1 \cup \{l\}, C_2 \cup \{\bar{l}\}}{C_1 \cup C_2}$$

$C_1 \cup C_2$  is the **resolvent** of the **parent clauses**  $C_1 \cup \{l\}$  and  $C_2 \cup \{\bar{l}\}$ .  $l$  and  $\bar{l}$  are the **resolution literals**.

**Example:**  $\{a, b, \neg c\}$  resolves with  $\{a, d, c\}$  to  $\{a, b, d\}$ .

**Note:** The resolvent is **not** logically equivalent to the set of parent clauses!

**Notation:**

$$R(\Delta) = \{C \mid C \text{ is resolvent of two clauses in } \Delta\}$$

## Resolution: Derivations

$D$  can be **derived** from  $\Delta$  by resolution (symbolically  $\Delta \vdash D$ ) if there is a sequence  $C_1, \dots, C_n$  of clauses such that

1.  $C_n = D$  and
2.  $C_i \in R(\Delta \cup \{C_1, \dots, C_{i-1}\})$ , for all  $i \in \{1, \dots, n\}$ .

Define  $R^*(\Delta) = \{D \mid \Delta \vdash D\}$ .

**Theorem (Soundness of resolution)**

Let  $D$  be a clause. If  $\Delta \vdash D$  then  $\Delta \models D$ .

**Proof idea.**

Show  $\Delta \models D$  if  $D \in R(\Delta)$  and use induction on proof length.

Let  $C_1 \cup \{l\}$  and  $C_2 \cup \{\bar{l}\}$  be the parent clauses of  $D = C_1 \cup C_2$ .

Assume  $\mathcal{I} \models \Delta$ , we have to show  $\mathcal{I} \models D$ .

Case 1:  $\mathcal{I} \models l$  then there must be a literal  $m \in C_2$  s.t.  $\mathcal{I} \models m$ . This implies  $\mathcal{I} \models D$ .

Case 2:  $\mathcal{I} \models \bar{l}$  similarly, there is  $m \in C_1$  s.t.  $\mathcal{I} \models m$ .

This means that each model  $\mathcal{I}$  of  $\Delta$  also satisfies  $D$ , i.e.,  $\Delta \models D$ . □

## Resolution: Completeness?

Do we have

$$\Delta \models \varphi \text{ implies } \Delta \vdash \varphi?$$

Of course, could only hold for CNF. However:

$$\left\{ \{a, b\}, \{\neg b, c\} \right\} \models \{a, b, c\}$$

$$\left\{ \{a, b\}, \{\neg b, c\} \right\} \not\models \{a, b, c\}$$

However, one can show that resolution is **refutation-complete**:

$$\Delta \text{ is unsatisfiable iff } \Delta \vdash \square.$$

**Entailment:** Reduce to unsatisfiability testing and decide by resolution.

## Resolution Strategies

- ▶ Trying out all different resolutions can be very costly,
- ▶ and might not be necessary.
- ▶ There are different **resolution strategies**.
- ▶ Examples:
  - ▶ **Input resolution** ( $R_I(\cdot)$ ): In each resolution step, one of the parent clauses must be a clause of the input set.
  - ▶ **Unit resolution** ( $R_U(\cdot)$ ): In each resolution step, one of the parent clauses must be a unit clause.
  - ▶ Not all strategies are (refutation) completeness preserving. Neither input nor unit resolution is. However, there are others.

## Horn Clauses & Resolution

**Horn clauses:** Clauses with at most one positive literal

**Example:**  $(a \vee \neg b \vee \neg c), (\neg b \vee \neg c)$

### Proposition

*Unit resolution is refutation-complete for Horn clauses.*

### Proof idea.

Consider  $R_U^*(\Delta)$  of Horn clause set  $\Delta$ . We have to show that if

$\square \notin R_U^*(\Delta)$ , then  $\Delta (\equiv R_U^*(\Delta))$  is satisfiable.

- ▶ Assign *true* to all unit clauses in  $R_U^*(\Delta)$ .
- ▶ Those clauses that do not contain a literal  $l$  such that  $\{l\}$  is one of the unit clauses have at least one negative literal.
- ▶ Assign true to these literals.
- ▶ Results in satisfying truth assignment for  $R_U^*(\Delta)$  (and  $\Delta \subseteq R_U^*(\Delta)$ ).

□