

Foundations of AI

8. Satisfiability and Model Construction

Davis-Putnam, Phase Transitions, GSAT

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Motivation

- Usually:
 - **Given:** A logical theory (set of propositions)
 - **Question:** Does a proposition **logically follow** from this theory?
 - Reduction to **unsatisfiability**, which is **coNP-complete** (complementary to NP problems)
- Sometimes:
 - **Given:** A logical theory
 - **Wanted:** **Model of the theory.**
 - **Example:** Configurations that fulfill the constraints given in the theory.
 - Can be “easier” because it is enough to find one model

The Davis-Putnam Procedure

DP Function

Given a set of clauses Δ defined over a set of variables Σ , return “satisfiable” if Δ is satisfiable. Otherwise return “unsatisfiable”.

1. If $\Delta = \emptyset$ return “satisfiable”
2. If $\square \in \Delta$ return “unsatisfiable”
3. **Unit-propagation Rule:** If Δ contains a **unit-clause** C , assign a truth-value to the variable in C that satisfies C , simplify Δ to Δ' and return **DP**(Δ').
4. **Splitting Rule:** Select from Σ a variable v which has not been assigned a truth-value. Assign one truth value t to it, simplify Δ to Δ' and call **DP**(Δ')
 - a. If the call returns “satisfiable”, then return “satisfiable”
 - b. Otherwise assign *the other* truth-value to v in Δ , simplify to Δ'' and return **DP**(Δ'').

Example (1)

$$\Delta = \{\{a, b, \neg c\}, \{\neg a, \neg b\}, \{c\}, \{a, \neg b\}\}$$

Example (2)

$$\Delta = \{\{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\}\}$$

Properties of DP

- DP is complete, correct, and guaranteed to terminate.
 - DP constructs a model, if one exists.
 - In general, DP requires **exponential time** (splitting rule!)
 - DP is **polynomial** on **horn clauses**, i.e., clauses with at most one positive literal.
 $(\neg A_1, \vee \dots \vee \neg A_n \vee B \equiv \wedge_i A_i \rightarrow B)$
- *Heuristics* are needed to determine which variable should be instantiated next and which value should be used
- In all SAT competitions so far, DP-based procedures have shown the best performance.

DP on Horn Clauses (1)

Note:

1. The simplifications in DP on Horn clauses always generate **Horn clauses**.
2. A set of Horn clauses **without unit clauses** is satisfiable
 - *All clauses have at least one negative literal*
 - *Assign false to all variables*
3. If the **first sequence of applications of the unit propagation rule** in DP does not lead to the empty clause, a set of Horn clauses without unit clauses is generated (which is satisfiable according to (2))

DP on Horn Clauses (2)

4. Although a set of Horn clauses without a unit clause is satisfiable, DP may **not immediately recognize** it.
 - a. If DP assigns *false* to a variable, this cannot lead to an unsatisfiable set and after a sequence of unit propagations we are in **the same situation as in 4**.
 - b. If DP assigns *true*, then we may get an empty clause - perhaps after unit propagation (and have to backtrack) - or the set is still satisfiable and we are in **the same situation as in 4**.

DP on Horn Clauses (3)

In summary:

1. DP executes a **sequence of unit propagation** steps resulting in
 - an empty clause or
 - a set of Horn clauses without a unit clause, which is satisfiable
 2. In the latter case, DP proceeds by **choosing** for one variable:
 - *false*, which does not change the satisfiability
 - *true*, which either
 - leads to an immediate contradiction (after unit propagation) and backtracking or
 - does not change satisfiability
- Run time is *polynomial* in the number of variables

How Good is DP in the Average Case?

- We know that SAT is NP-complete, i.e., in the worst case, it takes exponential time.
- This is clearly also true for the DP-procedure.
→ Couldn't we do better in the **average case**?
- For CNF-formulae in which the probability for a positive appearance, negative appearance and non-appearance in a clause is $1/3$, DP needs on average **quadratic time** (Goldberg 79)!
→ The probability that these formulae are satisfiable is, however, very high.

Phase Transitions ...

Conversely, we can, of course, try to identify **hard to solve** problem instances.

Cheeseman et al. (IJCAI-91) came up with the following plausible conjecture:

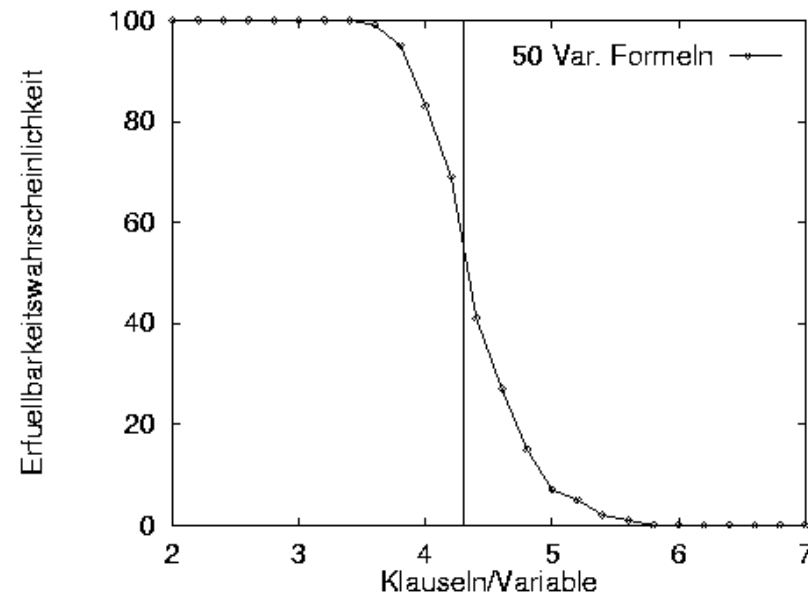
All NP-complete problems have at least *one order* parameter and the hard to solve problems are around a critical value of this order parameter. This critical value (a **phase transition**) separates one region from another, such as over-constrained and under-constrained regions of the problem space.

Confirmation for graph coloring and Hamilton path ... later also for other NP-complete problems.

Phase Transitions with 3-SAT

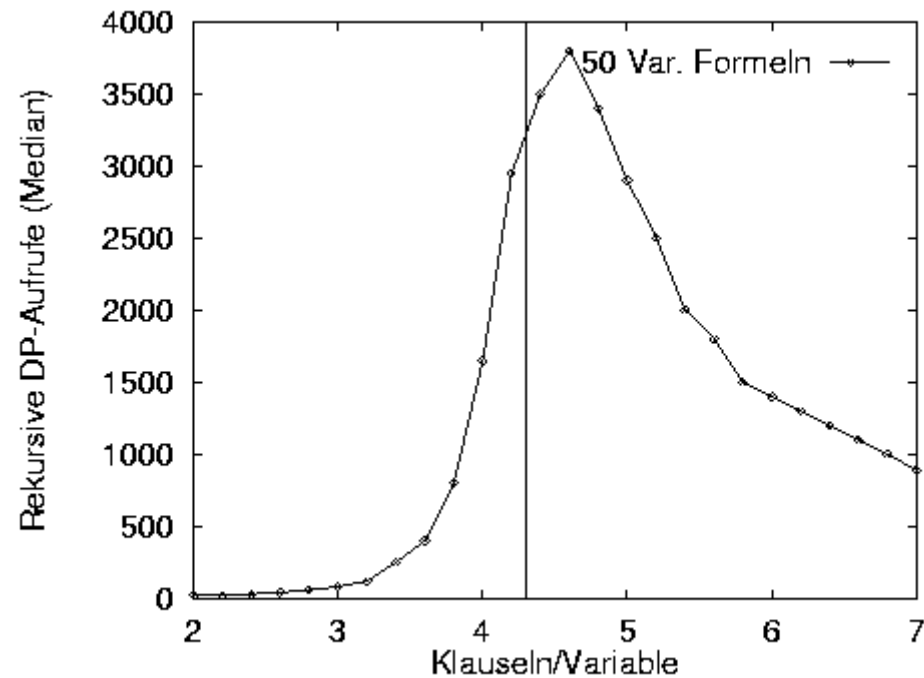
Constant clause length model (Mitchell et al., AAI-92):
Clause length k is given. Choose variables for every clause k and use the complement with probability 0.5 for each variable.

Phase transition for 3-SAT with a clause/variable ratio of approx. 4.3:



Empirical Difficulty

The Davis-Putnam (DP) Procedure shows extreme runtime peaks at the phase transition:



Note: Hard instances can exist even in the regions of the more easily satisfiable/unsatisfiable instances!

Notes on the Phase Transition

- When the probability of a solution is close to 1 (**under-constrained**), there are many solutions, and the first search path of a backtracking search is usually successful.
- If the probability of a solution is close to 0 (**over-constrained**), this fact can usually be determined early in the search.
- In the phase transition stage, there are many near successes (“close, but no cigar”).
 - (limited) possibility of predicting the difficulty of finding a solution based on the parameters.
 - (search intensive) benchmark problems are located in the phase region (but they have a special structure)

Local Search Methods for Solving Logical Problems

In many cases, we are interested in finding a satisfying assignment of variables (example CSP), and we can sacrifice completeness if we can “solve” much large instances this way.

Standard process for optimization problems:
Local Search

- Based on a (random) configuration
 - Through local modifications, we hope to produce better configurations
- Main problem: local maxima

Dealing with Local Maxima

As a measure of the value of a configuration in a logical problem, we could use the number of satisfied constraints/clauses.

But local search seems inappropriate, considering we want to find a global maximum (all constraints/clauses satisfied).

By **restarting** and/or **injecting noise**, we can often escape local maxima.

Actually: Local search performs very well for finding satisfying assignments of CNF formulae (even without injecting noise).

GSAT

Procedure GSAT

INPUT: a set of clauses α , MAX-FLIPS, and MAX-TRIES

OUTPUT: a satisfying truth assignment of α , if found

begin

 for i: =1 to MAX-TRIES

 T := a randomly-generated truth assignment

 for j: =1 to MAX-FLIPS

 if T satisfies α then return T

 v := a propositional variable such that a change in its truth assignment gives the largest increase in the number of clauses of α that are satisfied by T.

 T := T with the truth assignment of v reversed

 end for

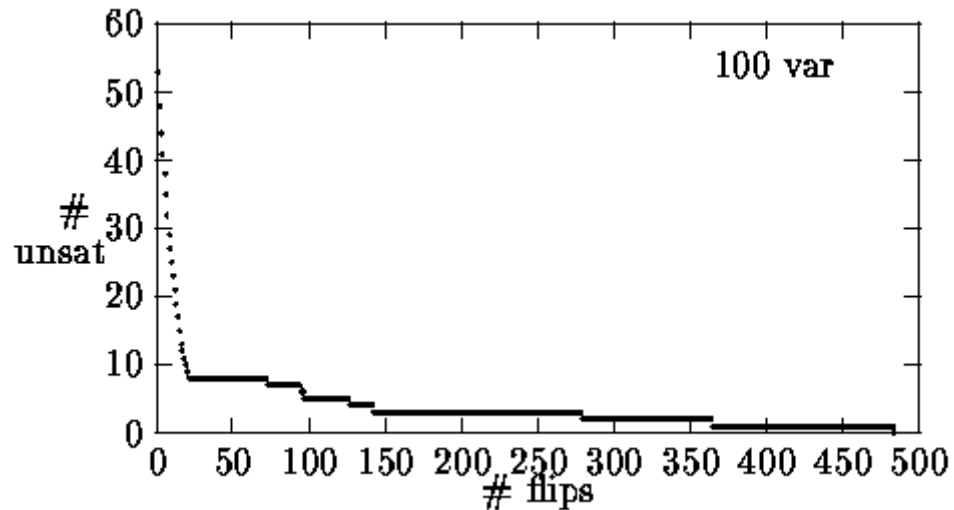
 end for

 return "no satisfying assignment found"

end

The Search Behavior of GSAT

- In contrast to normal local search methods, we must also allow sideways movements!
- Most time is spent searching on plateaus.



State of the Art

- SAT competitions since beginning of the '90
- Current SAT competitions (<http://www.satlive.org/>):
In 2003:
 - Largest solved instances:
100,000 variables / 1,000,000 clauses
 - Smallest unsolved instances:
200 variables/ 1,000 clauses
- Complete solvers are as good as randomized ones!

Concluding Remarks

- DP-based SAT solver prevail:
 - Very efficient implementation techniques
 - Good branching heuristics
 - Clause learning
- Incomplete randomized SAT-solvers
 - are good (in particular on random instances)
 - but there is no dramatic increase in size of what they can solve
 - parameters are difficult to adjust