Hoare Logic

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- standard presentation of Hoare logic:
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- here:
  invariants are given as part of correctness specification
- correctness proof possible only if invariants are adequate for
  pre- and postcondition pair

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- command

$$C \ ::= \ \textbf{skip} \mid x := e \mid C_1 \ ; \ C_2 \mid \textbf{if } b \textbf{ then } C_1 \textbf{ else } C_2 \mid \textbf{while } b \textbf{ do } C$$

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- interpretation of function symbol f in expression $f(e_1, \ldots, e_n)$
  depends on logical first-order model
  ("+" interpreted over model of unbounded integers or
  in model for modulo arithmetic?)

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- evaluation of Boolean expression b depends on logical first-order model
  ("$x \leq x + 1$" true in model of unbounded integers but false in model for modulo arithmetic)

Semantics of Commands C (1)

- semantics of command C
  = functions from set of states to set of states

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- execution of update statement
  = update of function $s : \textbf{Var} \to \textbf{Val}$

$$(x := e, s) \rightsquigarrow s' \quad \text{where} \quad \begin{array}{l} s'(x) = [\![e]\!](s) \;\; \text{and} \\ s'(y) = s(y) \;\; \text{for} \;\; x \not\equiv y \end{array}$$

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- execution of update depends on logical first-order model

Semantics of Commands C (2)

- execution of sequence of commands $C \equiv C_1 \,;\, C_2$
  $=$ execution of first command C_1 followed by execution of second command C_2

$$(C, s) \rightsquigarrow s'' \quad \text{if} \quad (C_1, s) \rightsquigarrow s' \quad \text{and} \quad (C_2, s') \rightsquigarrow s''$$

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- execution of command **skip** does not change state

$$(\textbf{skip}, s) \rightsquigarrow s$$

("empty sequence of commands")

Semantics of Commands C (3)

- execution of conditional command $C \equiv \textbf{if}\ b\ \textbf{then}\ C_1\ \textbf{else}\ C_2$ = execution of then-command C_1 if expression b evaluates to true

$$(C, s) \rightsquigarrow s' \quad \text{if} \quad [\![b]\!](s) = \mathbf{T} \quad \text{and} \quad (C_1, s) \rightsquigarrow s'$$

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- execution of conditional depends on logical first-order model

Semantics of Commands C (4)

- execution of while command $C \equiv$ **while** b **do** C_0
  $=$ execution of body C_0 followed by execution of while command C if expression b evaluates to true

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- execution of while loop depends on logical first-order model

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  for all states s
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  if $(C, s) \rightsquigarrow s'$ then
  $[\![\psi]\!](s') = \mathbf{T}$
- $\{\phi\}\ C\ \{\psi\}$ valid if valid in every logical first-order model
- $\Gamma \models \{\phi\}\ C\ \{\psi\}$ if $\{\phi\}\ C\ \{\psi\}$ valid in every logical first-order model of set of assertions Γ

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$$\textbf{if } x \leq y \textbf{ then skip else } z := y \ ;\ y := x \ ;\ x := z$$

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$$\textbf{if } x \leq y \textbf{ then skip else } z := y \ ; \ y := x \ ; \ x := z$$

- take precondition $\phi \ \equiv \ x = x_0 \wedge y = y_0 \wedge x_0 > y_0$
  and postcondition $\psi \ \equiv \ x = y_0 \wedge y = x_0$