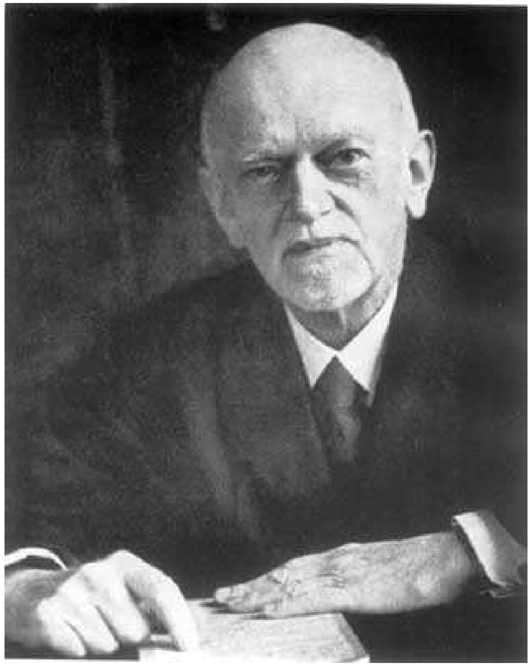
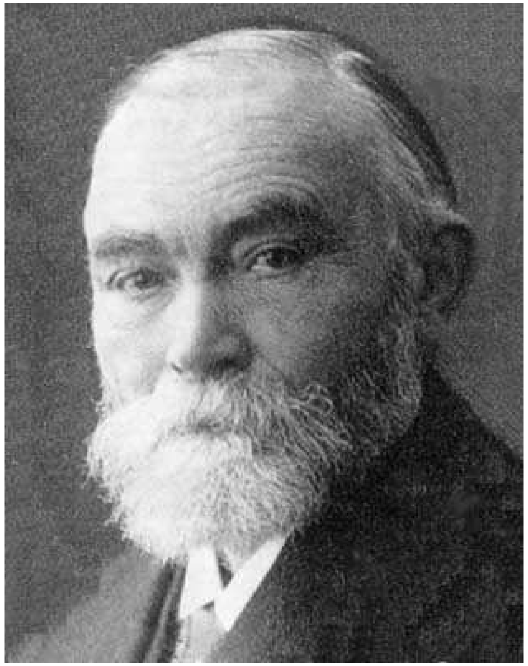
« Mathematical foundations: (2) Classical first-order logic »

George Boole



David Hilbert



Gottlob Frege

Reference

[1] Jean van Heijenoort, editor. “From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931”. Harvard University Press, 1967.

Formal logics

A formal logic consists of:

– a formal or informal language (formula expressing facts)

– a model-theoretic semantics (to define the meaning of the language, that is which facts are valid)

– a deductive system (made of axioms and inference rules to formaly derive theorems, that is facts that are provable)

Questions about formal logics

The main questions about a formal logic are:

– The soundness of the deductive system: no provable formula is invalid

– The completeness of the deductive system: all valid formulæ are provable

Propositional classical logic

Syntax of the classical propositional logic

Classical propositional logic

– $X \in \mathcal{V}$ are variables denoting unknown true or false facts
– The set of formulæ $\phi \in \mathcal{F}$ of the propositional logic are defined by the following grammar:

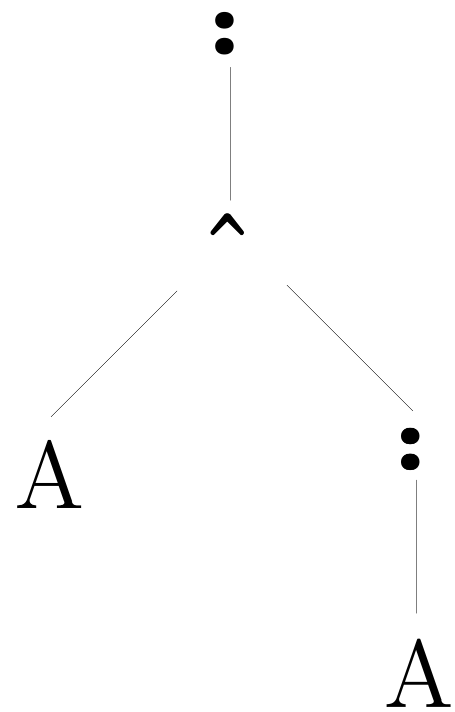
$$\begin{array}{rcl} \phi & ::= & X \\ & | & (\phi_1 \wedge \phi_2) \\ & | & (\neg \phi) \end{array}$$

– The relation "is a subformula of" is well founded, whence can be used for structural definitions and proofs

Example of formulæ

- A is a variable whence a formula
- $(\neg A)$ is a formula since A is a formula
- $(A \wedge (\neg A))$ is a formula since A and and $(\neg A)$ are formulae
- $(\neg(A \wedge (\neg A))$ is a formula since $(A \wedge (\neg A))$ is a formula

The derivation tree of the formula is:



Abstract syntax

– In practice we avoid parentheses thanks to priorities:

  - $\neg$ has highest priority (evaluated first)
  - $\wedge$ has lowest priority (evaluated second)
  - $\wedge$ is left associative (evaluation from left to right)

  For example, $\neg A \wedge \neg B \wedge C$ stands for $(\neg A) \wedge (\neg B) \wedge C$ which stands for $((\neg A) \wedge (\neg B)) \wedge C$

– The derivation tree is given by the following abstract grammar:

$$
\begin{array}{rcl}
\phi & ::= & X \\
 & \mid & \phi_1 \wedge \phi_2 \\
 & \mid & \neg \phi
\end{array}
$$

Propositional identities

Abbreviations (de Morgan laws)

$$\phi_1 \vee \phi_2 \stackrel{\text{def}}{=} \neg(\neg\phi_1 \wedge \neg\phi_2)$$

$$\phi_1 \Longrightarrow \phi_2 \stackrel{\text{def}}{=} \neg\phi_1 \vee \phi_2$$

$$\phi_1 \Longleftarrow \phi_2 \stackrel{\text{def}}{=} \phi_2 \Longrightarrow \phi_1$$

$$\phi_1 \Longleftrightarrow \phi_2 \stackrel{\text{def}}{=} (\phi_1 \Longrightarrow \phi_2) \wedge (\phi_1 \Longleftarrow \phi_2)$$

$$\phi_1 \veebar \phi_2 \stackrel{\text{def}}{=} (\phi_1 \vee \phi_2) \wedge \neg(\phi_1 \wedge \phi_2)$$

Free variables of proopositional formulae

The set $\mathrm{FV}(\phi)$ of free variables appearing in a formula ϕ is defined by structural induction as follows:

$$
\begin{aligned}
\mathrm{FV}(X) &\stackrel{\text{def}}{=} \{X\} \\
\mathrm{FV}(\neg\phi) &\stackrel{\text{def}}{=} \mathrm{FV}(\phi) \\
\mathrm{FV}(\phi_1 \wedge \phi_2) &\stackrel{\text{def}}{=} \mathrm{FV}(\phi_1) \cup \mathrm{FV}(\phi_2)
\end{aligned}
$$

Semantics of the propositional classical logic

Booleans

We define the booleans $\mathbb{B} \stackrel{\text{def}}{=} \{\mathtt{tt}, \mathtt{ff}\}$ and boolean operators by the following truth table:

$\overline{\&}$	$\mathtt{tt}$	$\mathtt{ff}$
$\mathtt{tt}$	$\mathtt{tt}$	$\mathtt{ff}$
$\mathtt{ff}$	$\mathtt{ff}$	$\mathtt{ff}$

$\overline{\neg}$	
$\mathtt{tt}$	$\mathtt{ff}$
$\mathtt{ff}$	$\mathtt{tt}$

Environment/Assignment

– An environment[1] $\rho \in \mathcal{V} \mapsto\!\!\!\!\!\not\;\; \mathbb{B}$ assigns boolean values $\rho(X)$ to free propositional variables X.

– An example of assignment is $\rho = \{X \to \mathtt{tt}, Y \to \mathtt{ff}\}$ such that $\rho(X) = \mathtt{tt}$, $\rho(Y) = \mathtt{ff}$ and the value for all other propositional variables $Z \in \mathcal{V} \setminus \{X, Y\}$ is undefined

[1] Also called *assignment* in logic.

Tarskian/model-theoretic semantics of the classical propositional logic

The semantics[2] $\mathcal{S} \in \mathcal{F} \mapsto (\mathcal{V} \mapsto \mathbb{B}) \mapsto \mathbb{B}$ of a propositional formula ϕ assign a meaning $\mathcal{S}[\![\phi]\!]\rho$ to the formula for any given environment ρ[3]:

$$
\begin{aligned}
\mathcal{S}[\![X]\!]\rho &\stackrel{\text{def}}{=} \rho(X) \\
\mathcal{S}[\![\neg\phi]\!]\rho &\stackrel{\text{def}}{=} \overline{\neg}(\mathcal{S}[\![\phi]\!]\rho) \\
\mathcal{S}[\![\phi_1 \wedge \phi_2]\!]\rho &\stackrel{\text{def}}{=} \mathcal{S}[\![\phi_1]\!]\rho \mathbin{\overline{\&}} \mathcal{S}[\![\phi_2]\!]\rho
\end{aligned}
$$

---

[2] Also called an *interpretation* in logic

[3] Hilbert used instead an arithmetic interpretation where 0 is true and 1 is false.

Models

ρ is a model of ϕ (or that ρ satisfies ϕ) if and only if:

$$\mathcal{S}[\![\phi]\!]\rho = \mathtt{tt}$$

which is written:

$$\rho \Vdash \phi$$

Entailment

– A set $\mathit{\Gamma} \in \wp(\mathcal{F})$ of formulae entails ϕ whenever:

$$\forall \rho : (\forall \phi' \in \mathit{\Gamma} : \rho \Vdash \phi') \Longrightarrow \rho \Vdash \phi$$

which is written:

$$\mathit{\Gamma} \Vdash \phi$$

Validity

– We say that ϕ is valid if and only if:

$$\forall \rho \in (\mathrm{V} \mapsto \mathbb{B}) : \mathcal{S}[\![\phi]\!]\rho = \mathtt{tt}$$

which is written:

$$\Vdash \phi$$

(i.e. ϕ is a tautaulogy, always true)

Examples of tautologies

$P \Longrightarrow P$

$(\neg\neg P) \Longrightarrow P$

$P \Longrightarrow (\neg\neg P)$

$P \Longrightarrow (Q \Longrightarrow P)$

$P \Longrightarrow (Q \Longrightarrow Q)$

$(\neg P \Longrightarrow P) \Longrightarrow P$

$P \Longrightarrow (\neg P \Longrightarrow Q)$

$\neg P \Longrightarrow (P \Longrightarrow Q)$

$(\neg(P \Longrightarrow P)) \Longrightarrow Q$

$P \Longrightarrow (\neg(P \Longrightarrow \neg P))$

$(P \Longrightarrow \neg P) \Longrightarrow \neg P$

$(\neg(P \Longrightarrow Q)) \Longrightarrow P$

$(\neg(P \Longrightarrow Q)) \Longrightarrow (\neg\neg P)$

$(\neg(P \Longrightarrow Q)) \Longrightarrow \neg Q$

$(P \Longrightarrow \neg P) \Longrightarrow (P \Longrightarrow Q)$

$(P \Longrightarrow Q) \Longrightarrow (\neg Q \Longrightarrow \neg P)$

$(P \Longrightarrow \neg Q) \Longrightarrow (Q \Longrightarrow \neg P)$

$(\neg P \Longrightarrow \neg Q) \Longrightarrow (Q \Longrightarrow P)$

$(\neg P \Longrightarrow \neg Q) \Longrightarrow (\neg P \Longrightarrow Q) \Longrightarrow P)$

$(\neg(P \Longrightarrow Q)) \Longrightarrow (Q \Longrightarrow R)$

$(\neg(P \Longrightarrow Q)) \Longrightarrow (\neg P \Longrightarrow R)$

$(P \Longrightarrow Q) \Longrightarrow ((Q \Longrightarrow R) \Longrightarrow (P \Longrightarrow R))$

Satisfiability/Unsatisfiability

– A formula $\phi \in \mathcal{F}$ is satisfiable if and only if:

$$\exists \rho \in (\mathrm{V} \mapsto \mathbb{B}) : \mathcal{S}[\![\phi]\!]\rho = \mathtt{tt}$$

– A formula $\phi \in \mathcal{F}$ is unsatisfiable if and only if:

$$\forall \rho \in (\mathrm{V} \mapsto \mathbb{B}) : \mathcal{S}[\![\phi]\!]\rho = \mathtt{tt}$$

(i.e. ϕ is a antilogy, always false)

Satisfiability/Validity/Unsatisfiability

Formulae

Satisfiable

Sometimes true
Sometimes false

Valid

Always true

Unsatisfiable

Always false

Deductive system for the classical propositional logic

Hilbert deductive system

– Axiom schemata[4]:

(1) $\phi \vee \phi \Longrightarrow \phi$[5]

(2) $\phi \Longrightarrow \phi' \vee \phi$[6]

(3) $(\phi \Longrightarrow \phi') \Longrightarrow (\phi'' \vee \phi \Longrightarrow \phi' \vee \phi'')$[7]

– Inference rule schema[4]:

$$(\mathrm{MP}) \quad \frac{\phi, \ \phi \Longrightarrow \phi'}{\phi'}$$[8] modus ponens

[4] to be instanciated for all possible formulae $\phi, \phi', \phi'' \in \mathcal{F}$

[5] i.e. $\neg(\neg(\neg\phi \wedge \neg\phi)) \vee \phi)$

[6] i.e. $\neg(\neg\neg\phi \wedge \neg\neg(\neg\phi \wedge \neg\phi'))$

[7] i.e; $\neg(\neg\phi \vee \phi') \vee (\neg(\phi'' \vee \phi) \vee (\phi' \vee \phi''))$ where $\phi_1 \vee \phi_2 \stackrel{\mathrm{def}}{=} \neg(\neg\phi_1 \vee \neg\phi_2)$

[8] i.e. $\dfrac{\phi, \ \neg\phi \vee \phi'}{\phi'}$

Hilbert derivation

– A derivation from a set $\Gamma \in \wp(\mathcal{F})$ of hypotheses is a finite nonempty sequence:

$$\phi_1, \phi_2, \ldots, \phi_n \qquad\qquad n \geq 0$$

of formulae such that for each ϕ_i, $i = 1, \ldots, n$, we have:

- ϕ_i is a element of Γ (hypothesis)
- ϕ_i is an axiom
- ϕ_i is the conclusion of an inference rule $\dfrac{\phi_i^1, \ldots, \phi_i^k}{\phi_i}$ such that $\{\phi_i^1, \ldots, \phi_i^k\} \subseteq \{\phi_1, \phi_2, \ldots, \phi_{n-1}\}$ [9]

[9] So that the premises have already been proved.

Hilbert proof

– A proof is a derivation from $\emptyset$

Example of proof

$$
\begin{array}{lll}
 & (\phi \vee \phi \Longrightarrow \phi) \Longrightarrow (\neg\phi \vee (\phi \vee \phi) \Longrightarrow \phi \vee \neg\phi) & \\
 & & \wr[\text{instance of } (3)]\ (a)\wr \\
 & \phi \vee \phi \Longrightarrow \phi & \wr[\text{instance of } (1)]\ (b)\wr \\
 & \neg\phi \vee (\phi \vee \phi) \Longrightarrow (\phi \vee \neg\phi) & \wr[(a), (b) \text{ and } (MP)]\ (c)\wr \\
= & (\phi \Longrightarrow (\phi \vee \phi)) \Longrightarrow \phi \vee \neg\phi & \wr\text{def.} \Longrightarrow \text{abbreviation}\wr \\
 & \phi \Longrightarrow (\phi \vee \phi) & \wr[\text{instance of } (2)]\ (d)\wr \\
 & \phi \vee \neg\phi & \wr[(c), (d) \text{ and } (MP)]\wr
\end{array}
$$

Hilbert provability

- $\phi \in \mathrm{F}$ is provable from $\Gamma \in \wp(\mathrm{F})$ (or Γ proves ϕ) iff there is a proof of ϕ from Γ, written:

$$\Gamma \vdash \phi$$

where the deduction system (axioms and inference rules) are understood from the context.

- $\emptyset \vdash \phi$ is written $\vdash \phi$

This is the proof-theoretic semantics of first-order logic.

Example of provability

$$\vdash \neg\phi \vee \neg\neg\phi$$

PROOF.
Replace ϕ by $\neg\phi$ is the previous proof of $\phi \vee \neg\phi$.
$\square$

<u>Soundness</u> of a deductive system

Provable formulae do hold:

$$\Gamma \vdash \phi \Longrightarrow \Gamma \Vdash \phi$$

Proof.
The proof for propositional logic is by induction on the length of the formal proof of ϕ from Γ.

A proof of length one, can only use a formula ϕ in Γ which is assumed to hold (i.e. $\mathcal{S}[\![\phi]\!]\rho = \mathtt{tt}$) or an axiom that does hold as shown below.

– $\mathcal{S}[\![\phi \vee \phi \Longrightarrow \phi]\!]\rho$

$$
\begin{array}{ll}
= \mathcal{S}[\![\neg(\neg(\neg\phi \wedge \neg\phi))]\!]\rho & \text{def. } \vee \\
= \overline{\neg}(\overline{\neg}(\overline{\neg}(\mathcal{S}[\![\phi]\!]\rho)\,\overline{\&}\,\overline{\neg}(\mathcal{S}[\![\phi]\!]\rho))) & \text{def. } \mathcal{S} \\
= \overline{\neg}(\mathcal{S}[\![\phi]\!]\rho)\,\overline{\&}\,\overline{\neg}(\mathcal{S}[\![\phi]\!]\rho) & \text{def. } \overline{\neg} \\
= \overline{\neg}(\mathtt{ff}) & \text{def. } \overline{\&}
\end{array}
$$

$= \mathtt{tt}$ def. $\overline{\neg}$

– The proof is similar for the other two axioms.

A proof of length $n+1$, $n \geq 1$ is an initial proof $\phi_0, \ldots, \phi_{n-1}$ of length n followed by a formula ϕ_n. By induction hypothesis, we have $\mathcal{S}[\![\phi_i]\!]\rho = \mathtt{tt}$, $i = 1, \ldots, n-1$.

If $\phi_n \in \mathit{\Gamma}$ or ϕ_n is an axiom then $\mathcal{S}[\![\phi_n]\!]\rho = \mathtt{tt}$ as shown above.

Otherwise, ϕ_n is derived by the modus ponens inference rule (MP). In that case, we have k, $0 \leq k < n$ such that $\mathcal{S}[\![\phi_k]\!]\rho = \mathtt{tt}$ and $\mathcal{S}[\![\phi_k \Longrightarrow \phi_n]\!]\rho = \mathtt{tt}$ so $(\mathcal{S}[\![\phi_k]\!]\rho \overline{\Longrightarrow} \mathcal{S}[\![\phi_n]\!]\rho) = \mathtt{tt}$ where the truth table of $\overline{\Longrightarrow}$ is derived from the definition of $\Longrightarrow$ and that of $\overline{\neg}$ and $\overline{\wedge}$ as follows:

$\overline{\Longrightarrow}$	$\mathtt{ff}$	$\mathtt{tt}$
$\mathtt{ff}$	$\mathtt{tt}$	$\mathtt{tt}$
$\mathtt{tt}$	$\mathtt{ff}$	$\mathtt{tt}$

Since $\mathcal{S}[\![\phi_k]\!]\rho = \mathtt{tt}$ the truth table of $\overline{\Longrightarrow}$ shows than the only possibility for $(\mathcal{S}[\![\phi_k]\!]\rho \overline{\Longrightarrow} \mathcal{S}[\![\phi_n]\!]\rho) = \mathtt{tt}$ is $\mathcal{S}[\![\phi_n]\!]\rho = \mathtt{tt}$. $\square$

Consistency of a deductive system

Absence of contradictory proofs

$$\overline{\neg}(\exists \Gamma : \Gamma \vdash \phi \mathrel{\overline{\wedge}} \Gamma \vdash \neg\phi)$$

A sound deductive system is consistent.

PROOF.
By reduction ad absurdum assume inconsistency $\exists \Gamma : \Gamma \vdash \phi \mathrel{\overline{\wedge}} \Gamma \vdash \neg\phi$. By soundness $\Gamma \Vdash \phi \mathrel{\overline{\wedge}} \Gamma \Vdash \neg\phi$ whence for all ρ such that $\forall \phi' \in \Gamma : \rho \Vdash \phi'$, we have $\mathcal{S}[\![\phi]\!]\rho = \mathit{tt}$ and $\mathcal{S}[\![\neg\phi]\!]\rho = \mathit{tt} = \overline{\neg}\mathcal{S}[\![\phi]\!]\rho = \overline{\neg}\mathit{tt} = \mathit{ff}$ which is the desired contradiction since $\mathit{tt} \neq \mathit{ff}$. $\square$

Negative normal form

A formula is in negative normal form iff it can be parsed by the following grammar:

$$
\begin{aligned}
\phi &::= \phi \vee \phi \\
&\quad\mid\ \phi \wedge \phi \\
&\quad\mid\ \varphi \\
\varphi &::= X \\
&\quad\mid\ \neg X
\end{aligned}
$$

Normalization in negative normal form

$$
\begin{aligned}
\mathrm{nnf}(\neg\phi) &\stackrel{\mathrm{def}}{=} \overline{\mathrm{nnf}}(\phi) \\
\mathrm{nnf}(\phi_1 \vee \phi_2) &\stackrel{\mathrm{def}}{=} \mathrm{nnf}(\phi_1) \vee \mathrm{nnf}(\phi_2) \\
\mathrm{nnf}(\phi_1 \wedge \phi_2) &\stackrel{\mathrm{def}}{=} \mathrm{nnf}(\phi_1) \wedge \mathrm{nnf}(\phi_2) \\
\overline{\mathrm{nnf}}(\neg\phi) &\stackrel{\mathrm{def}}{=} \mathrm{nnf}(\phi) \\
\overline{\mathrm{nnf}}(\phi_1 \vee \phi_2) &\stackrel{\mathrm{def}}{=} \overline{\mathrm{nnf}}(\phi_1) \wedge \overline{\mathrm{nnf}}(\phi_2) \\
\overline{\mathrm{nnf}}(\phi_1 \wedge \phi_2) &\stackrel{\mathrm{def}}{=} \overline{\mathrm{nnf}}(\phi_1) \vee \overline{\mathrm{nnf}}(\phi_2) \\
\mathrm{nnf}(X) &\stackrel{\mathrm{def}}{=} X \\
\overline{\mathrm{nnf}}(X) &\stackrel{\mathrm{def}}{=} \neg X
\end{aligned}
$$

A formula ϕ is equivalent to its negative normal form $\mathrm{nnf}(\phi)$ is that:

$$\vdash \phi \quad \text{if and only if} \ \vdash \mathrm{nnf}(\phi)$$

Conjunctive normal form

A formula is in conjunctive normal form iff it can be parsed by the following grammar:

$$
\begin{aligned}
\phi &::= \phi^{\wedge} \\
\phi^{\wedge} &::= \phi^{\wedge} \wedge \phi^{\wedge} \\
&\mid \quad \phi^{\vee} \\
\phi^{\vee} &::= \phi^{\vee} \vee \phi^{\vee} \\
&\mid \quad \varphi \\
\varphi &::= X \\
&\mid \quad \neg X
\end{aligned}
$$

Normalization in conjunctive normal form

Any formula ϕ can be put in equivalent conjunctive normal form by applying the following transformations to $\mathrm{nnf}(\phi)$:

$$\phi' \vee (\phi_1 \wedge \phi_2) \rightsquigarrow (\phi' \wedge \phi_1) \vee (\phi' \wedge \phi_2)$$
$$(\phi_1 \vee \phi_2) \wedge \phi' \rightsquigarrow (\phi_1 \vee \phi') \wedge (\phi_2 \vee \phi')$$

A formula ϕ is equivalent to its conjunctive normal form $\phi^\wedge$ in that:

$$\vdash \phi \quad \text{if and only if} \quad \vdash \phi^\wedge$$

Completeness of a deductive system

Formulae which hold are provable:

$$\Gamma \Vdash \phi \Longrightarrow \Gamma \vdash \phi$$

The very first proof for propositional logic was given by Bernays (a student of Hilbert) [2]. The better known proof is that of Post [3].

Reference

[2] Richard Zach. "Completeness before Post: Bernays, Hilbert, and the development of propositional logic", Bulletin of Symbolic Logic 5 (1999) 331–366.

[3] Ryan Stansifer. "Completeness of Propositional Logic as a Program", Florida Institute of Technology, Melbourne, Florida, March 2001.

Bernay's proof can be sketched as follows. Every formula is interderivable with its conjunctive normal form. A conjuction is provable if and only if each of its conjuncts is provable. A disjunction of propositional variables or negations of proprositional variables if and only if it contains a variable and its negation, and conversely, every such disjunction is provable. So a formula is provable if and only if every conjunct in its normal form contains a variable and its negation. Now suppose that ϕ is a valid ($\Vdash \phi$) but underivable formula. Its conjunctive normal form $\phi^\wedge$ is also underivable, so it must contain a conjunct ϕ' where every variable occurs only negated or unnegated but not both. If ϕ where added as a new axiom (so that $\Vdash \phi$ implies soundness of the new deductive system), then $\phi^\wedge$ and ϕ' would also be derivable. By substituting X for every unnegated variable and $(\neg X)$ for every negated variable in ϕ', we would obtain X as a derivable formula (after some simplification), and the system would be inconsistent, which is the desired contradiction.

Classical first-order logic

Syntax of the classical first-order logic

Lexems

The lexems are the basic constituants of the formal language.

– symbols: $(, \; ,, \;), \; \wedge, \; \neg, \; \forall, \; \ldots$

– constants: $a, b \ldots \in \mathcal{C}$ denote individual objects of the universe of discourse

– variables: $x, y, \ldots \in \mathcal{V}$ denote unknown but fixed [10] objects of the universe of discourse

[10] Different instances of the same variable in a given scope of a formula always denote the same unkown individal object of the universe of discourse. This is not true of imperative computer programs.

- function symbols: $f\backslash n, g\backslash n, \ldots \in \mathcal{F}^n$ denote fonctions of arity n. We let $\mathcal{F}^0 \stackrel{\text{def}}{=} \mathcal{C}$ and $\mathcal{F} = \bigcup_{n\in\mathbb{N}} F^n$. For short we write f instead of $f\backslash n$ when the arity n is understood
- relation symbols: $r\backslash n, \rho\backslash n, \ldots \in \mathcal{R}^n$ denote fonctions of arity n. We let $\mathbb{B} \stackrel{\text{def}}{=} \{\text{tt}, \text{ff}\}$ and $\mathcal{R} = \bigcup_{n\in\mathbb{N}} R^n$. For short we write r instead of $r\backslash n$ when the arity n is understood

Terms

Terms $t \in \mathcal{T}$ denote individual objects of the universe of discourse computed by applying fonctions to constants or variables:

$$
\begin{aligned}
t ::= \; & c \\
\mid \; & x \\
\mid \; & f\backslash n(t_1, \ldots, t_n)
\end{aligned}
$$

Atomic formulæ

Atomic formulæ $A \in \mathcal{A}$ are used to state elementary facts about objects of the universe of discourse:

$$A ::= r\backslash n(t_1, \ldots, t_n)$$

Example:

– z is a variable whence a term

– $*\backslash 2(+\backslash 2(x, 1), y)$ is a term

– $\leq\backslash 2$ is a relation symbol whence $\leq\backslash 2(*\backslash 2(+\backslash 2(x, 1), y), z)$ [11] is an atomic formula

11 written $((x + 1) * y) \leq z$ in infix form

First-order formulae

The set $\Phi \in \mathcal{L}$ of first-order formulae (of the first-order language $\mathcal{L}$) is defined by the following grammar

$$
\begin{array}{rlll}
\Phi ::= & A & & A \in \mathcal{A} \\
\mid & \forall x : \Phi & & x \in \mathcal{V} \\
\mid & \Phi_1 \vee \Phi_2 & & \\
\mid & \neg \Phi & &
\end{array}
$$

$\exists x : \Phi$ is a shorthand for $\neg(\forall x : (\neg \Phi))$

Bound variables

Bound variables appear under the scope of a quantifier:

$$
\begin{aligned}
\mathrm{bv}(\forall x : \Phi) &\stackrel{\mathrm{def}}{=} \{x\} \cup \mathrm{bv}(\Phi)\\
\mathrm{bv}(\Phi_1 \vee \Phi_2) &\stackrel{\mathrm{def}}{=} \mathrm{bv}(\Phi_1) \cup \mathrm{bv}(\Phi_2)\\
\mathrm{bv}(\neg\Phi) &\stackrel{\mathrm{def}}{=} \mathrm{bv}(\Phi)\\
\mathrm{bv}(r\backslash n(t_1, \ldots, t_n)) &\stackrel{\mathrm{def}}{=} \emptyset\\
\mathrm{bv}(c) &\stackrel{\mathrm{def}}{=} \emptyset\\
\mathrm{bv}(x) &\stackrel{\mathrm{def}}{=} \emptyset\\
\mathrm{bv}(f\backslash n(t_1, \ldots, t_n)) &\stackrel{\mathrm{def}}{=} \emptyset
\end{aligned}
$$

Free variables

Free variables are not bound by a quantifier:

$$\begin{aligned}
\mathrm{fv}(\forall x : \Phi) &\stackrel{\mathrm{def}}{=} \mathrm{fv}(\Phi) \setminus \{x\} \\
\mathrm{fv}(\Phi_1 \vee \Phi_2) &\stackrel{\mathrm{def}}{=} \mathrm{fv}(\Phi_1) \cup \mathrm{fv}(\Phi_2) \\
\mathrm{fv}(\neg \Phi) &\stackrel{\mathrm{def}}{=} \mathrm{fv}(\Phi) \\
\mathrm{fv}(r \backslash n(t_1, \ldots, t_n)) &\stackrel{\mathrm{def}}{=} \bigcup_{i=1}^{n} \mathrm{fv}(t_i) \\
\mathrm{fv}(c) &\stackrel{\mathrm{def}}{=} \emptyset \\
\mathrm{fv}(x) &\stackrel{\mathrm{def}}{=} \{x\} \\
\mathrm{fv}(f \backslash n(t_1, \ldots, t_n) &\stackrel{\mathrm{def}}{=} \bigcup_{i=1}^{n} \mathrm{fv}(t_i)
\end{aligned}$$

Theories

- The set of variables of a formula is $\mathrm{var}(\Phi) \stackrel{\mathrm{def}}{=} \mathrm{bv}(\Phi) \cup \mathrm{fv}(\Phi)$
- A closed sentence (or ground formula) is a formula Φ with no free variable (so that $\mathrm{fv}(\Phi) = \emptyset$)
- A theory is a set of closed sentences

Substitution

– Substitution is a syntactic replacement of a variable by a term, may be with appropriate renaming of bound variables, so as to avoid capturing the term free variables, as in

$$\begin{array}{ll} & \exists x : x = y + 1[y := x] \\ \not\to & \exists x : x = x + 1 \end{array}$$

but should be

$$\to \quad \exists x' : x' = x + 1$$

A substitution $\sigma \in \mathcal{V} \mapsto \mathcal{T}$ is a function from variables to terms with finite domain:

$$
\begin{array}{rcll}
\mathrm{dom}(\sigma) & \stackrel{\mathrm{def}}{=} & \{x \in \mathcal{V} \mid x \neq \sigma(x)\} & \text{(finite domain)} \\
\mathrm{rng}(\sigma) & \stackrel{\mathrm{def}}{=} & \{\sigma(x) \mid x \in \mathrm{dom}(\sigma)\} & \text{(range)} \\
\mathrm{yld}(\sigma) & \stackrel{\mathrm{def}}{=} & \bigcup\{\mathrm{fv}(t) \mid t \in \mathrm{rng}(\sigma)\} & \text{(yield)}
\end{array}
$$

We write σ as:

$$[x_1 \leftarrow \sigma(x_1), \ldots, x_n \leftarrow \sigma(x_n)]$$

where $\mathrm{dom}(\sigma) = \{x_1, \ldots, x_n\}$.

Application of a substitution to a term

$$
\begin{aligned}
\sigma(c) &\stackrel{\text{def}}{=} c \\
\sigma(y) &\stackrel{\text{def}}{=} y & \text{iff } y \notin \operatorname{dom}(\sigma) \\
\sigma(f(t_1, \ldots, t_n)) &\stackrel{\text{def}}{=} f(\sigma(t_1), \ldots, \sigma(t_n)) \\
\sigma(r(t_1, \ldots, t_n)) &\stackrel{\text{def}}{=} r(\sigma(t_1), \ldots, \sigma(t_n)) \\
\sigma(\neg \Phi) &\stackrel{\text{def}}{=} \neg \sigma(\Phi) \\
\sigma(\Phi_1 \vee \Phi_2) &\stackrel{\text{def}}{=} \sigma(\Phi_1) \vee \sigma(\Phi_2) \\
\sigma(\forall x : \Phi) &\stackrel{\text{def}}{=} \forall x' : \sigma(\Phi[x := x']) & \text{where} \\
& \qquad x' \notin \operatorname{yld}(\sigma) \cup (\operatorname{fv}(\Phi) \setminus \{x\})
\end{aligned}
$$

Example of substitution in a term

$$
\begin{array}{rl}
& (\exists x : x = y + 1)[y := x] \\
= & \exists x' : ((x = y + 1)[x := x'])[y := x] \\
= & \exists x' : ((x)[x := x'] = (y)[x := x'] + (1)[x := x'])[y := x] \\
= & \exists x' : (x' = y + 1)[y := x] \\
= & \exists x' : ((x')[y := x] = (y)[y := x] + (1)[y := x]) \\
= & \exists x' : ((x')[y := x] = (y)[y := x] + (1)[y := x])
\end{array}
$$

Semantics of the classical first-order logic

Interpretation

An interpretation I is defined by:

- A domain of discourse D_I (or domain of interpretation)
- An interpretation $I[\![f]\!] \in D_I^m \mapsto D_I$ for each function symbol $f \in \mathcal{F}^m$, $m \geq 0$ (including constants)
- An interpretation $I[\![r]\!] \in D_I^m \mapsto \mathbb{B}$ for each relation symbol $r \in \mathcal{R}^m$, $m \geq 0$

Environment/Assignment

– An environment/assignment $\rho \in \mathcal{V} \mapsto D_I$ assigns a value $\rho(x)$ to each variable $x \in \mathcal{V}$

Assignment notation: if $f \in A \mapsto B$, $a \in A$, $b \in B$ then $f[a := b] = f' \in A \mapsto B$ such that:

$f'(a) = b$ i.e. $f[a := b](a) = b$

$f'(x) = f(x)$ whenever $x \neq a$ i.e. $f[a := b](x) = f(x)$

Semantics of the first-order logic

Given an interpretation I, the semantics is:

$$\mathcal{S}^I[\![t]\!] \in (\mathcal{V} \mapsto D_I) \mapsto D_I$$

$$\begin{aligned}
\mathcal{S}^I[\![c]\!]\rho &\stackrel{\text{def}}{=} I[\![c]\!] \\
\mathcal{S}^I[\![x]\!]\rho &\stackrel{\text{def}}{=} \rho(x) \\
\mathcal{S}^I[\![f(t_1, \ldots, t_n)]\!]\rho &\stackrel{\text{def}}{=} I[\![f]\!](\mathcal{S}^I[\![t_1]\!]\rho, \ldots, \mathcal{S}^I[\![t_n]\!]\rho)
\end{aligned}$$

$$\mathcal{S}^I[\![A]\!] \in (\mathcal{V} \mapsto D_I) \mapsto \mathbb{B}$$

$$\mathcal{S}^I[\![r(t_1, \ldots, t_n)]\!]\rho \stackrel{\text{def}}{=} I[\![r]\!](\mathcal{S}^I[\![t_1]\!]\rho, \ldots, \mathcal{S}^I[\![t_n]\!]\rho)$$

$$\mathcal{S}^I[\![\Phi]\!] \in (\mathcal{V} \mapsto D_I) \mapsto \mathbb{B}$$

$$\mathcal{S}^I[\![\neg\Phi]\!]\rho \stackrel{\text{def}}{=} \overline{\neg}(\mathcal{S}^I[\![g\Phi]\!]\rho)$$

$$\mathcal{S}^I[\![\Phi_1 \vee \Phi_2]\!]\rho \stackrel{\text{def}}{=} \mathcal{S}^I[\![\Phi_1]\!]\rho \mathbin{\overline{\vee}} \mathcal{S}^I[\![\Phi_2]\!]\rho$$

$$\mathcal{S}^I[\![\forall x : \Phi]\!]\rho \stackrel{\text{def}}{=} \overline{\bigwedge_{v \in D_I}}^{12} \mathcal{S}^I[\![\Phi]\!]\rho[x := v]$$

---

12 If $S \subseteq \mathbb{B}$ then $\overline{\bigwedge} S \stackrel{\text{def}}{=} (S \subseteq \{\mathtt{tt}\})$.

It follows that for the abbreviations, we have:

$$\mathcal{S}^I[\![\Phi_1 \Longrightarrow \Phi_2]\!]\rho \stackrel{\text{def}}{=} \mathcal{S}^I[\![\Phi_1]\!]\rho \;\overline{\Longrightarrow}\; \mathcal{S}^I[\![\Phi_2]\!]\rho$$

$$\mathcal{S}^I[\![\exists x : \Phi]\!]\rho \stackrel{\text{def}}{=} \overline{\bigvee_{v \in D_I}} \mathcal{S}^I[\![\Phi]\!]\rho[x := v]$$

where:

$\overline{\Longrightarrow}$	ff	tt
ff	tt	tt
tt	ff	tt

$\overline{\vee}$	ff	tt
ff	ff	tt
tt	tt	tt

and if $S \subseteq \mathbb{B}$ then $\overline{\bigvee} S \stackrel{\text{def}}{=} (S \cap \{\text{tt}\} \neq \emptyset)$.

Semantics of substitution

Assignment is the semantic counterpart of syntactic substitution:

$$\mathcal{S}^I[\![\sigma(\Phi)]\!]\rho = \mathcal{S}^I[\![\Phi]\!]\rho'$$
$$\text{where} \quad \forall x \in \mathcal{V} : \rho'(x) = \mathcal{S}^I[\![\sigma(x)]\!]\rho$$

Lemma

If $x \notin \text{fv}[\![t]\!]$ then

$$\forall \rho \in \mathcal{V} \mapsto D_I : \forall v \in D_I : \mathcal{S}^I[\![t]\!]\rho = \mathcal{S}^I[\![t]\!]\rho[x := v]$$

Proof.

- The case $t = x$ is disallowed by $x \notin \text{fv}[\![x]\!] = \{x\}$
- If $y \neq x$ then $x \notin \text{fv}[\![y]\!] = \{y\}$ and $\mathcal{S}^I[\![y]\!]\rho = \rho(y) = \rho[x := v](y) = \mathcal{S}^I[\![y]\!]\rho[x := v]$
- $\mathcal{S}^I[\![f(t_1, \ldots, t_n)]\!]\rho$
  $= I[\![f]\!](\mathcal{S}^I[\![t_1]\!]\rho, \ldots, \mathcal{S}^I[\![t_n]\!]\rho)$
  $= I[\![f]\!](\mathcal{S}^I[\![t_1]\!]\rho[x := v], \ldots, \mathcal{S}^I[\![t_n]\!]\rho[x := v])$ by induction hypothesis since $\forall i : x \notin \text{fv}[\![t_i]\!]$
  $= \mathcal{S}^I[\![f(t_1, \ldots, t_n)]\!]\rho[x := v]$
- $\mathcal{S}^I[\![r(t_1, \ldots, t_n)]\!]\rho$
  $= I[\![r]\!](\mathcal{S}^I[\![t_1]\!]\rho, \ldots, \mathcal{S}^I[\![t_n]\!]\rho)$

$= I[\![r]\!](\mathcal{S}^I[\![t_1]\!]\rho[x := v], \ldots, \mathcal{S}^I[\![t_n]\!]\rho[x := v])$ by induction hypothesis since $\forall i : x \not\in \mathrm{fv}[\![t_i]\!]$

$= \mathcal{S}^I[\![r(t_1, \ldots, t_n)]\!]\rho[x := v]$

$\square$

Proof of the theorem

Proof.

By structural induction on formulae

- $\mathcal{S}^I[\![\sigma(c)]\!]\rho = \mathcal{S}^I[\![c]\!]\rho = I[\![c]\!] = \mathcal{S}^I[\![c]\!]\rho'$
- $\mathcal{S}^I[\![\sigma(x)]\!]\rho = \rho'(x) = \mathcal{S}^I[\![x]\!]\rho'$
- $\mathcal{S}^I[\![\sigma(f(t_1, \ldots, t_n))]\!]\rho$
  $= \mathcal{S}^I[\![f(\sigma(t_1), \ldots, \sigma(t_n))]\!]\rho$
  $= I[\![f]\!](\mathcal{S}^I[\![\sigma(t_1)]\!]\rho, \ldots, \mathcal{S}^I[\![\sigma(\sigma(t_n)]\!]\rho)$
  $= I[\![f]\!](\mathcal{S}^I[\![t_1]\!]\rho', \ldots, \mathcal{S}^I[\![t_n]\!]\rho'$
  $= \mathcal{S}^I[\![f(t_1, \ldots, t_n)]\!]\rho')$

proving that $\forall t : \mathcal{S}^I[\![\sigma(t)]\!]\rho \ = \ \mathcal{S}^I[\![t]\!]\rho'$

- $\mathcal{S}^I[\![\sigma(r(t_1, \ldots, t_n))]\!]\rho$
  $= \mathcal{S}^I[\![r(\sigma(t_1), \ldots, \sigma(t_n))]\!]\rho$
  $= I[\![r]\!](\mathcal{S}^I[\![\sigma(t_1)]\!]\rho, \ldots, \mathcal{S}^I[\![\sigma(\sigma(t_n)]\!]\rho)$
  $= I[\![r]\!](\mathcal{S}^I[\![t_1]\!]\rho', \ldots, \mathcal{S}^I[\![t_n]\!]\rho'$

$= \mathcal{S}^I[\![r(t_1, \ldots, t_n)]\!]\rho')$

proving that $\forall A : \mathcal{S}^I[\![\sigma(A)]\!]\rho \;=\; \mathcal{S}^I[\![A]\!]\rho'$

- $\mathcal{S}^I[\![\sigma(\neg\Phi)]\!]\rho = \mathcal{S}^I[\![\neg\sigma(\Phi)]\!]\rho = \overline{\neg}(\mathcal{S}^I[\![\sigma(\Phi)]\!]\rho) = \overline{\neg}(\mathcal{S}^I[\![\Phi]\!]\rho') = \mathcal{S}^I[\![\neg\Phi]\!]\rho'$
- $\mathcal{S}^I[\![\sigma(\Phi_1 \vee \Phi_2)]\!]\rho = \mathcal{S}^I[\![\sigma(\Phi_1) \vee \sigma(\Phi_2)]\!]\rho = \mathcal{S}^I[\![\sigma(\Phi_1)]\!]\rho \overline{\vee} \mathcal{S}^I[\![\sigma(\Phi_2)]\!]\rho = \mathcal{S}^I[\![\Phi_1]\!]\rho' \overline{\vee} \mathcal{S}^I[\![\Phi_2]\!]\rho' = \mathcal{S}^I[\![\Phi_1 \vee \Phi_2]\!]\rho'$
- $\mathcal{S}^I[\![\sigma(\forall x : \Phi)]\!]\rho$

$$
\begin{aligned}
&= \mathcal{S}^I[\![\forall x' : \sigma(\Phi[x := x'])]\!]\rho \\
&\qquad \wr \text{where } x' \notin \mathrm{yld}(\sigma) \cup (\mathrm{fv}(\Phi) \setminus \{x\}) \int \\
&= \mathcal{S}^I[\![\forall x' : \sigma([x \leftarrow x']\,^{13}(\Phi))]\!]\rho \\
&= \mathcal{S}^I[\![\forall x' : (\sigma \circ [x \leftarrow x'])\,^{14}(\Phi))]\!]\rho \\
&= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![(\sigma \circ [x \leftarrow x'])(\Phi))]\!]\rho[x' := v] \\
&= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![\phi]\!] \left(\lambda y \cdot \mathcal{S}^I[\![(\sigma \circ [x \leftarrow x'])(y))]\!]\rho[x' := v]\right) \;\wr \text{by induction hypothesis} \int
\end{aligned}
$$

$$= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![\phi]\!](\lambda y \cdot (\!| y = x \mathbin{?} \mathcal{S}^I[\![(\sigma \circ [x \leftarrow x'])(y))]\!]\rho[x' := v] \mathbin{¿} \mathcal{S}^I[\![(\sigma \circ [x \leftarrow x'])(y))]\!]\rho[x' := v] |\!)^{15})$$

$$= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![\phi]\!](\lambda y \cdot (\!| y = x \mathbin{?} \mathcal{S}^I[\![\sigma(x')]\!]\rho[x' := v] \mathbin{¿} \mathcal{S}^I[\![\sigma(y)]\!]\rho[x' := v] |\!))$$

$$= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![\phi]\!](\lambda y \cdot (\!| y = x \mathbin{?} \mathcal{S}^I[\![x']\!]\rho[x' := v] \mathbin{¿} \mathcal{S}^I[\![\sigma(y)]\!]\rho[x' := v] |\!))$$

$\wr$since $x' \notin \mathrm{yld}(\sigma)$ so that $\sigma(x') = x'\int$

$$= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![\phi]\!](\lambda y \cdot (\!| y = x \mathbin{?} v \mathbin{¿} \mathcal{S}^I[\![y]\!]\rho' |\!))$$

$\wr$since

- $\mathcal{S}^I[\![x']\!]\rho[x' := v] = \rho[x' := v](x') = v$
- $x' \notin \mathrm{yld}(\sigma)$ so that $x' \in \mathrm{fv}[\![\sigma(y)]\!]$ hence, by the lemma, $\mathcal{S}^I[\![\sigma(y)]\!]\rho[x' := v] = \mathcal{S}^I[\![\sigma(y)]\!]\rho = \mathcal{S}^I[\![y]\!]\rho'$ by induction hypothesis

$\int$

$$= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![\phi]\!](\lambda y \cdot (\!|\, y = x \;?\; v \;:\; \rho'(y) |\!))$$

$$= \overline{\bigwedge_{v \in D_I}} \mathcal{S}^I[\![\phi]\!](\rho'[x := v])$$

$$= \mathcal{S}^I[\![\forall x : \phi]\!]\rho'$$

□

13 The function $[x \leftarrow x']$ is the substitution of x' for x

14 $\circ$ is function composition $f \div \mathit{compg}(x) \stackrel{\text{def}}{=} f(g(x))$

15 The conditional is $(\!|\, \mathtt{tt} \;?\; a : b |\!) = a$ and $(\!|\, \mathtt{ff} \;?\; a : b |\!) = b$ and $(\!|\, a \;?\; b \,\|\, c \;?\; d : e |\!) = (\!|\, a \;?\; b : (\!|\, c \;?\; d : e |\!) |\!)$

Deductive system for the classical first-order logic

Deduction system for first-order logic (H)

– Axioms (for all instances of formulae Φ, Φ', Φ', variable x and term t):

$$
\begin{array}{ll}
(1) & \Phi \vee \Phi \Longrightarrow \Phi \\
(2) & \Phi \Longrightarrow \Phi' \vee \Phi \\
(3) & (\Phi \Longrightarrow \Phi') \Longrightarrow (\Phi'' \vee \Phi \Longrightarrow \Phi' \vee \Phi'') \\
(4) & \forall x : \Phi \Longrightarrow \Phi[x := t] \\
(5) & (\forall x : \Phi \vee \Phi') \Longrightarrow \Phi \vee \forall x : \Phi' \text{ when } x \notin \mathrm{fv}(\Phi)
\end{array}
$$

– Inference rules (for all instances of formulae Φ, Φ' and variable x):

$$\text{(MP)} \qquad \frac{\Phi, \ \Phi \Longrightarrow \Phi'}{\Phi'} \qquad \text{Modus Ponens}$$

$$\text{(Gen)} \qquad \frac{\Phi}{\forall x : \Phi} \qquad \text{Generalization}$$

Example 1 of proof

$$\Phi[x := t] \Longrightarrow \neg\forall x : \neg\Phi \quad (\text{i.e. } \exists x : \Phi)$$

PROOF. (assuming tautologies for short)

(a) $\forall x : \neg\Phi \Longrightarrow (\neg\Phi)[x := t]$ $\wr$instance of (4)$\wr$

(b) $(\Phi \Longrightarrow \Phi') \Longrightarrow (\neg\Phi' \Longrightarrow \neg\Phi)$ $\wr$contraposition tautology$\wr$

(b') $(\forall x : \neg\Phi \Longrightarrow (\neg\Phi)[x := t]) \Longrightarrow \neg((\neg\Phi)[x := t]) \Longrightarrow \neg\forall x : \neg\Phi$ $\wr$tautology, instance of (b)$\wr$

(c) $\neg((\neg\Phi)[x := t]) \Longrightarrow \neg\forall x : \neg\Phi$ $\wr$(a), (b') and (MP)$\wr$

(c') $\neg\neg(\Phi[x := t]) \Longrightarrow \neg\forall x : \neg\Phi$ $\wr$def. substitution$\wr$

(d) $(\neg\neg\Phi \Longrightarrow \neg\Phi') \Longrightarrow (\Phi \Longrightarrow \neg\Phi')$ $\wr$tautology$\wr$

(d') $(\neg\neg(\Phi[x := t]) \Longrightarrow \neg\forall x : \neg\Phi) \Longrightarrow (\Phi[x := t] \Longrightarrow \neg\forall x : \neg\Phi)$ $\wr$tautology$\wr$

(e) $\Phi[x := t] \Longrightarrow \neg\forall x : \neg\Phi$ $\wr$(c), (d') and (MP)$\wr$

$\square$

Example 2 of proof

$$\{\Phi \Longrightarrow \Phi'\} \vdash \neg\forall x : \neg\Phi \Longrightarrow \Phi' \quad \text{when } x \notin \text{fv}(\Phi')$$

PROOF. (assuming tautologies for short)

(a)	$\Phi \Longrightarrow \Phi'$	$\wr$hypothesis$\wr$
(b)	$(\Phi \Longrightarrow \Phi') \Longrightarrow (\neg\Phi' \Longrightarrow \neg\Phi)$	$\wr$contraposition tautology$\wr$
(c)	$\neg\Phi' \Longrightarrow \neg\Phi$	$\wr$(a), (b) and (MP)$\wr$
(c')	$\neg\neg\Phi' \vee \neg\Phi$	$\wr$def. abbreviation $\Longrightarrow\wr$
(d)	$\forall x : (\neg\neg\Phi' \vee \neg\Phi)$	$\wr$(c'), (Gen)$\wr$
(e)	$\neg\neg\Phi' \vee \forall x : \neg\Phi$	$\wr$(d), (5), $x \notin \text{fv}(\neg\neg\Phi') = \text{fv}(\Phi')\wr$
(f)	$\neg\Phi' \Longrightarrow \forall x : \neg\Phi$	$\wr$def. abbreviation $\Longrightarrow\wr$
(g)	$(\neg\Phi' \Longrightarrow \forall x : \neg\Phi) \Longrightarrow (\neg\forall x : \neg\Phi \Longrightarrow \neg\neg\Phi')$	$\wr$contraposition tautology$\wr$
(h)	$\neg\forall x : \neg\Phi \Longrightarrow \neg\neg\Phi'$	$\wr$(f), (g) and (MP)$\wr$

(i) $(\Phi \Longrightarrow \neg\neg\Phi') \Longrightarrow (\Phi \Longrightarrow \Phi')$ $\wr$tautology$\int$

(i') $(\neg\forall x : \neg\Phi \Longrightarrow \neg\neg\Phi') \Longrightarrow (\neg\forall x : \neg\Phi \Longrightarrow \Phi')$ $\wr$tautology, instance of (i)$\int$

(j) $\neg\forall x : \neg\Phi \Longrightarrow \Phi'$ $\wr$(h), (i') and (MP)$\int$

$\square$

Extension of the deduction system (H) for first-order logic

These theorems are often incorporated to the deductive system as an axiom

$$\Phi[x := t] \Longrightarrow \exists x : \Phi$$

and a generalization rule:

$$\frac{\Phi \Longrightarrow \Phi'}{(\exists x : \Phi) \Longrightarrow \Phi'} \text{ when } x \notin \mathrm{fv}(\Phi)$$

Logical equivalences involving quantifiers and negations

- $\neg\forall x : \Phi \iff \exists x : \neg Phi$ De Morgan laws
- $\neg\exists x : \Phi \iff \forall x : \neg\Phi$
- $(\forall x : \Phi \wedge \forall x : \Phi) \iff \forall x : (\Phi \wedge \Phi')$
- $(\exists x : \Phi \vee \forall x : \Phi) \iff \exists x : (\Phi \vee \Phi')$
- $(\Phi \Longrightarrow \Phi') \Longrightarrow (\exists x : \Phi \Longrightarrow \Phi')$ when $x \notin \mathrm{fv}(\Phi')$
- $(\Phi \Longrightarrow \Phi') \Longrightarrow (\Phi \Longrightarrow \forall x : \Phi')$ when $x \notin \mathrm{fv}(\Phi')$
- $\forall x : (\Phi \vee \Phi') \iff (\forall x : \Phi) \vee \Phi'$ when $x \notin \mathrm{fv}(\Phi')$
- $\exists x : (\Phi \wedge \Phi') \iff (\exists x : \Phi) \wedge \Phi'$ when $x \notin \mathrm{fv}(\Phi')$
- $\Phi \iff \forall x : \Phi$ when $x \notin \mathrm{fv}(\Phi')$
- $\Phi \iff \exists x : \Phi$ when $x \notin \mathrm{fv}(\Phi')$

Properties of the deduction system (H) for first-order logic

– The Hilbert style deductive system (H) is sound, consistent, compact [16] and complete [4] for the first-order-logic.

Reference

[4] Kurt Gödel. "Die Vollständigkeit der Axiome des logischen Funktionen-kalküls", *Monatshefte für Mathematik und Physik* 37 (1930), 349-360.

16 $\Gamma \vdash \Phi$ if and only if $\Gamma' \vdash \Phi$ for a finite subset Γ' of Γ.

- The Hilbert style deductive system (H) is not decidable [5].
- Proofs cannot be fully automated: there is no terminating algorithm that, given a first-order formula Φ as input, returns true whenever Φ is classically valid.

Reference

[5] Kurt Gödel. "Über Formal Unentscheidbare Sätze der Principia Mathematica und Verwandter Systeme, I". *Monatshefte für Mathematik und Physik* 38, 173–198, 1931.

The theory axiomatizing equality

Writing $= \backslash 2(A, B)$ as $A = B$, the theory axiomatizing equality is first-order logic plus the following axioms:

- $\forall x : x = x$ reflexivity
- $\forall x : \forall y : (x = y) \Longrightarrow (y = x)$ symmetry
- $\forall x_1 : \ldots \forall x_n : \forall y_1 : \ldots \forall y_n : (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \Longrightarrow (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$ Leibnitz functional congruence
- $\forall x_1 : \ldots \forall x_n : \forall y_1 : \ldots \forall y_n : (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \Longrightarrow (r(x_1, \ldots, x_n) = r(y_1, \ldots, y_n))$ Leibnitz relational congruence
- $\forall x : \forall y : \forall z : (x = y \wedge y = z) \Longrightarrow (x = z)$ transitivity

Peano arithmetic [6]

- Constant symbols: 0
- Functional symbols: s (sucessor), $+$, $\times$
- Relation symbols: $=$, $\leq$
- Axioms:
  - $\forall x : x = x$ reflexivity
  - $\forall x : \forall y : (x = y) \Longrightarrow (y = x)$ symmetry
  - $\forall x : \forall y : \forall z : (x = y \wedge y = z) \Longrightarrow (x = z)$ transitivity

- $\forall x : \forall y : (x = y) \Longrightarrow (s(x) = s(y))$ congruence
- $\forall x : \forall y : \forall z : \forall t : (x = z \wedge t = t) \Longrightarrow (x + y = z + t)$
- $\forall x : \forall y : \forall z : \forall t : (x = z \wedge t = t) \Longrightarrow (x \times y = z \times t)$
- $\forall x : \forall y : \forall z : \forall t : (x = z \wedge t = t) \Longrightarrow (x \leq y = z \leq t)$
- $\forall x : (x = 0) \vee (\exists y : x = s(y))$ every natural but 0 is a successor
- $\forall x : \neg(s(x) = 0)$ 0 is not a successor
- $\forall x : \forall y : (s(x) = s(y)) \Longrightarrow (x = y)$ s is injective so every nonzero natural has a unique predecessor
- $\forall x : x + 0 = x$ def. addition
- $\forall x : \forall y : s + s(y) = s(x + y)$

- $\forall x : x \times 0 = 0$ def. multiplication
- $\forall x : \forall y : x \times s(y) = (x \times y) + x$
- $((\Phi)[x := 0] \wedge (\forall x : \Phi \Longrightarrow (\Phi)[x := s(x)]) \Longrightarrow (\forall x : \Phi)$ recurrence (for all instances of Φ)

Reference

[6] Giuseppe Peano. Arithmetices principia, nova methodo exposita. Augustae Taurinorum, Ed. Fratres Bocca, 1889. – XVI, 20 p.

Non standard integers

This axiomatization formalizes natural numbers but does not exclude "non standard models" of the form:

$0\ 1\ 2\ 3\ \ldots\ \ \ldots -2^0\ -1^0\ 0^0\ 1^0\ 2^0\ \ldots\ \ -2^1\ -1^1\ 0^1\ 1^1\ 2^1\ \ldots\ \ -2^2\ -1^2\ 0^2\ 1^2\ 2^2\ \ldots\ \ \ldots$

Excluded by the second-order logic induction axiom [17]:

$$\forall P : (P(0) \wedge (\forall x : P(x) \Longrightarrow P(s(x)))) \Longrightarrow \forall x : P$$

[17] The difference is that there is a denumerable infinity of instances of Φ while there can be a non-denumerable infinity of Ps, see G.S.Boolos and R.C.Jeffrey, "Computability and Logic", Cambridge University Press, 1974, 1980, 1989, Section 17, pp.193-195.

THE END