Lorenzen's Games and Linear Logic

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Abstract

This paper presents some basic notions concerning Lorenzen's game tradition and its relation with Linear Logic. It is shown that Lorenzen's idea was applied in Blass' work in order to define a game semantics for Linear Logic. Originally Blass came-up with these ideas on 70's, however they were fully developed only in 1992. Moreover, by identifying the source of the problems in Blass' approach, Abramsky developed a system using the same pattern of games for which Linear Logic is sound and complete. These systems are both exposed in the following text. Finally, some further issues are discussed and an alternative (and intuitive) approach is built in order to achieve the desired semantics taking into account finite games instead of infinite.

1 Lorenzen's Game Semantics

Each logic has a syntactical aspect (which defines the well-formed constructions) and a semantical aspect, in which the meaning of each construction is set in an unique way. In classical logic the latter is given by truth values associated with its compound elements (atoms). Therefore, to ascertain which conditions make a certain formula be valid (or not valid), one has to assign every possible valuation to these atoms, namely, to build its truth table.

Yet, intuitionistic logic has a different standpoint with respect to semantics. In this logic one has to have a proof of a formula to determine whether the formula is valid or not, i.e. knows how to construct a proof of the formula.

From intuitionistic logic Lorenzen built a *new* method to assevere the meaning of a formula. Instead of truth-values or proofs, Lorenzen considers a semantics in which the meaning of a formula is drained from dialogues in the sense that a formula is debated until both parts reach an agreement, meaning one side has no more arguments to expose and thus he has to agree and accept what is being given.

In order to define the proposed semantics, two characters are introduced: (*i*)A proponent (called P), who wants to verify the formula that he himself has proposed as valid, and (*ii*) a opponent (O) who has the role of denying the formula which was proposed by proponent. Henceforth P stands for proponent and O for opponent. Moreover, the words "*debate*", "*game*" and "*dialogue*" are used synonymously.

So the system sketched by Lorenzen includes the above mentioned players P and O, the propositions (which are interpreted as games), connectives (which are operations on games) and validity which is established by the existence of a winning strategy for P in the proposed formula). Further, each connective is associated with an argumentation rule which describes how the debate should proceed with respect to each connective. Table 1 describes the argumentation method for the set of basic connectives and for quantifiers in accordance with Felscher [Fel86] and van Benthem [Ben98].

$\wedge$: P asserts $\psi \wedge \gamma$ **Attack**: O chooses one conjunct **Defense**: P Asserts that conjunct	$\vee$: P asserts $\psi \vee \gamma$ **Attack**: O asks one disjunct **Defense**: P asserts and defends one disjunct
$\rightarrow$: P asserts $\psi \rightarrow \gamma$ **Attack**: O attacks ψ **Defense**: P defends γ	$\neg$: P asserts $\neg\psi$ **Attack**: O attacks ψ **Defense**: No defense available
$\forall$: P asserts $\forall x.P(x)$ **Attack**: O chooses one x **Defense**: P asserts that x is s.t. $P(x)$	$\exists$: P asserts $\exists x.P(x)$ **Attack**: O attacks $\exists$ **Defense**: P has to show one x is s.t. $P(x)$

Table 1: Lorenzen's rules.

A dialogue is defined, then, as a finite sequence of statements the are made alternately by P and O, following the rules above. An extra definition that is required concerns the starting player – it must be always P.

The player P is declared the winner of a dialogue when he has made the last move in such a way that O has no reply to it. On the other hand, O is the winner in the case when he has made the last move and P has nothing to do.

Lorenzen's intentions were to encompass a semantics for constructive logic, however the previous set of rules (table 1) was not strong enough to capture the whole idea. Therefore, Lorenzen extended this set by adding four rules. Further, a dialogue that follows both sets of rules is called a *D-dialogue*. The previously mentioned rules are the following:

1. P may assert an atomic formula only after it has been asserted by O;
2. If there are more than one attacks left (open) to be answered by P, then the only one that can be answered is the most recent;
3. An attack must be answered at most once;
4. An assertion made by P may be attacked at most once.

To understand how a Lorenzen's game style works and to show how important are the four rules above the following, comparison is suitable. Suppose "$\neg(\neg\psi \wedge \psi)$" and "$\psi \vee \neg\psi$" and describe a game for these formulas in accordance to the set of rules defined in table 2.

The difference in these two debates is that the rules concerning re-attacking and re-defenses allow the distinction between these formulas in such a way that if one drops the four rules out, the above dialogues would have the same winner. Furthermore the dialogue won by P is constructively valid while the dialogue won by O it is not.

The previous set of rules define what is called by Felscher as a *D-dialogue* scheme so that it enables Felscher to state the following theorem:

Theorem 1 (Lorenzen's D-dialogue conversion) *There exist recursive algorithms which for every formula v transform a proof of the sequent $\Rightarrow v$ in Gentzen's LJ (for intuitionistic logic) into a D-strategy.*

1.	P:	$\neg(\psi \wedge \neg\psi)$		$\psi \vee \neg\psi$	
2.	O:	$\psi \wedge \neg\psi$	[A,1]	$\vee$	[A,1]
3.	P:	$?L$	[A,2]	$\neg\psi$	[D,2]
4.	O:	ψ	[D,3]	ψ	[A,3]
5.	P:	$?R$	[A,1]		
6.	0:	$\neg\psi$	[D,5]		
7.	P:	ψ	[A,6]		

Table 2: Description of two games

The real proof was not given by Lorenzen, on the contrary, this framework become discredited due the lack of proofs.

In order to prove this theorem, Felscher inputs one more rule and calls the system following such rules as *E-dialogues*. The additional rule is the following:

5. O can react only upon the immediately preceding claim of P.

And with the addition of the rule above, it is possible to give the *Extension Lemma*, which is a essential tool to prove the theorem above. The lemma is as follows:

Lemma 1.1 (Extension Lemma) *There is a recursive algorithm by which every E-strategy can be embedded into a D-strategy.*

Once all this machinery has been defined, the idea may take a step further used as a background to another logic. Next section covers Andreas Blass' attempt to use Lorenzen's game semantics style in linear logic.

2 Blass' Approach

From Lorenzen, Blass[1] took the idea – the constructive meaning of a proposition ψ should be given by telling how to conduct a dialogue between a proponent P who want to assert ψ and an opponent O who wants to deny it.

From Girard [Gir87] [Gir93], Blass got the underlying logic - linear logic, the "resource-conscious logic"[Tro92], that is, a logic in which handling of resources is explicitly taken into account. Thus, propositions are interpreted as games, connectives as operation on games and validity as the existence of a winning strategy for P.

In order to combine both ideas using games, the first change provided by Blass [Bla92] concerns the length of a dialogue which is now allowed to take infinitely many moves. Specifically, an atomic formula is not an end point to a debate, on the contrary, it may be debatable like any other formula. Further, Blass interprets it as a method to point-out the situation when both players do not have a winning strategy, i.e. neither P nor O knows whether a formula is valid or not. Should be noted at this point that the "infinite games concept" makes the first rule established by Lorenzen redundant in this context.

Blass also provides a second change in Lorenzen's concept of games and this change concerns the rules about re-attacking and re-defending. Blass simply took these rules out and and replaced them in the original system with a set of connectives regarding these issues. These connectives are natural implementations of the previous rules so that the original system gains a new conjunction and a new disjunction. The rules on table 3 make these distinctions clear.

The winner of this kind of game is defined as follows:

- P wins a debate on $\psi \otimes \gamma$ if he wins both ψ-sub-debate and γ-sub-debate;

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[1] Despite the fact that Blass' ideas are mostly linked up with linear logic, it is worth noting that his fundamental notions were raised on the early 70's and do not regard linear logicsince this logical approach was developed on the late 80's by Girard

- P wins a debate on $\psi\widetilde{\otimes}\gamma$ if he wins at least one of these sub-debates.

By adding an unary symbol ranging over unlimited repetition and by defining constants to the new connectives introduced above, Blass attains linear logic. A game for unlimited repetition follows the same pattern as a series of $\otimes$-games while the constants $1, \top$ stand for games which P always wins and $0, \perp$ for games which O always wins.

Formally, Blass defines games as an ordered triple (M, s, G) where M stands for the set of possible moves, s stands for the information about who is the starting player and G is the set of games won by P, i.e. infinite sequences of members of M (moves). Following the terminology given by Lorenzen, Blass defines a position in a game as a finite sequence of moves and a strategy for both players P and O as a function τ, which $Ran(\tau) \subset M$, from the set of all positions where the player is to move. Further this strategy is considered a winning strategy to P if all plays in which P follows τ are in G. Otherwise, if none of the moves are in G the winning strategy is for O. A game as the one defined by Blass does not need to be determined since it is allowed to have infinite length, i.e. it is possible that neither P nor O has a winning strategy.

Once the formal structure of a game has been defined, linear logic's connectives can be interpreted as operations on the structure presented above. The following specifies this in detail:

A *negation* of a game is the simple change of roles between P and O, which is represented on the previously defined structures as:

$$\neg(M, s, G) \equiv (M, \bar{s}, M^{\star} \perp G)$$

The *standard conjunction* $(\wedge)$, namely *additive conjunction* – inherited from classical logic and intuitionistic logic – of two games is defined as follows:

$$(M_0, s_0, G_0) \wedge (M_1, s_1, G_1)$$

$\otimes$: Two debates: one for each conjunct $\psi \wedge \gamma$	$\widetilde{\otimes}$: Two debates: one for each disjunct:
Assertion: P asserts $\psi \otimes \gamma$	**Assertion**: P asserts $\psi\widetilde{\otimes}\gamma$
Attack: O chooses the starting sub-debate (ψ or γ) (if it is the first move), continue in a debate or choose another conjunct.	**Attack**: O demands P to choose
Defense: P asserts one of the conjuncts chosen by O.	**Defense**: P asserts one disjunct (if it is the first move), resume in that disjunct or switch to another disjunct

Table 3: Description of multiplicative connectives behaviour.

This game has $M_0 \cup M_1 \cup \{0,1\}$[2] as the set of moves and it starts with O choosing either 0 or 1. This choice selects the set of rules G_{choice} that is going to be followed. P has a winning strategy in this game only if he holds a winning strategy for both conjuncts.

For the standard disjunctions one has to consider the change of roles, so that:

$$(M_0, s_0, G_0)\vee(M_1, s_1, G_1)$$

And the formal definition of $\vee$ can be copied from the additive conjunction's definition by exchanging P by O (and vice-versa) in every occurence of these terms.

The relative simplicity of these games is blurred by the rules concerning non-standard disjunction and conjunction, also called *multiplicative conjunction* or *tensor product* and *multiplicative disjunction* or *par* in linear logic.

To the multiplicative conjunction, consider that M_0 and M_1, in G_0 and G_1 respectively, are disjoint sets of moves. So the set of moves is $M_0 \cup M_1 \cup \{0,1\}$. If a position P is a sequence of moves, $(p)_0$ and $(p)_1$ are subsequences of P of moves in M_0 and M_1. In any position P, O is to move if and only if O is to move in both $(p)_0$ and $(p)_1$, according to G_0 and G_1, and P is to move at any position P in $G_0 \otimes G_1$ if and only if he is to move at $(p)_0$ or $(p)_1$ or in both.

A play for this conjunction has O as winner if at least one of the subsequences of $(\chi)_i$ is in the complement of the set of games won by P, i.e. if it is infinite and if it is a win for O on the game G_i.

As happens to standard disjunction, the multiplicative disjunction is defined using the same dual scheme and the only change to be applied concerns the interchange of roles of P and O. So:

$$(G_0 \widetilde{\otimes} G_1) \equiv \neg(\neg G_0 \otimes \neg G_1)$$

The last connective that Blass defines is the unary repetition, i.e. unlimited repetition of a certain game G. At a position P, O is to move in $R(G)$ if and only if O is to move at all the positions $(p)_i$ in G. Conversely, P is to move in $R(G)$ at any position P if and only if for some i, P is to move in G at $(p)_i$. An interesting issue pointed out by Blass must be spotted-out here – P must play consistently in all constituent games so that P has to follow the games played by O. This is the distinguishing factor between $R(G)$ and tensor product

The dual operator of $R(G)$ is $\widetilde{R}(G)$ and it follows the same dual scheme definition. Therefore the following is valid:

$$\widetilde{R}(G) \equiv \neg R(\neg G)$$

And again, to obtain a formal definition of this connective, one has just to exchange P by O in R's definition.

Lastly, Blass defined the constants for multiplicative and additive disjunction and conjunction. To these constants one player always has a winning strategy. In the case of $\top$, P always has a winning strategy and in $\bot$'s case P has no way to win a game at all. Further Blass defined $\top$ to be the game with set of moves

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[2] This extension needs to be inserted in order to allow the conjunct's/disjunct's choice.

$M = \{O\}$ with just one move executed by O and the only possible play is a win for P. The definition of $\perp$ is exactly the same, the only thing that should be noted is the interchange of roles.

This completes the linear logic semantics definition. However, Blass noticed that the definition he achieved was, in fact, affine logic (linear logic plus contraction's rule). The reason for that relies in the fact that tensor product fails to be complete in this interpretation for what it is concluded that Girard's proposed interpretation is stronger (with respect to the proof rules) to Blass' game definition.

The following section exposes Abramsky's approach to game semantics and linear logic. Furthermore, the following presents a complete interpretation using a categorical model of linear logic in which formulas are objects and proof are morphisms.

3 Abramsky's Approach

Abramsky suggests a game semantics for linear logic in which formulas denote games and proofs stand for winning strategies. This framework leads to a categorical model in which it is possible to prove full completeness for multiplicative linear logic with MIX-rule (MLL+MIX) so that every strategy is the denotation of a unique cut-free proof-net. This semantics has a history-free concept as underlying idea and this shows itself strongly connected with geometry of interaction.

3.1 Games and strategies

In defining games, Abramsky does not explicitly mention Lorenzen, however it can be noted that the same concept is used in order to have a proper notion of games. Each game involves two players with opposite roles – Player[3] and Opponent. A *play* (or *run*) of a game is an alternate sequence of moves, which can be either finite or infinite. Every run has a winner, that is, each run can be either a win for P or for O. Further, by convention, the first move in a run is always executed by O.

In a short comparison to Blass, one can see the structure representing games as simpler than the structure proposed by Abramsky. However, it is not the case – both of these structures carry the same amount of information but in the case of Abramsky's, the extra component can also be extracted in Blass' with a simple operation over sets. Abramsky defines the game structure for a game A as follows:

$$A = (M_A, \lambda_A, P_A, W_A)$$

where

M_A is the set of moves;

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[3] Called *proponent* in Blass sense. (To avoid confusion we follow Blass' nomenclature).

Game ≡ Process specification
Moves ≡ Alphabet or sorts of actions
Proponent ≡ System
Opponent ≡ Environment
P_a ≡ Safety specification
W_a ≡ Liveness specification
Strategy ≡ Process
Strategy in A ≡ Process satisfying liveness specification
Winning strategy ≡ Deadlock-free process satisfying liveness specification

Table 4: Relation between game's concept and process theory

> $\lambda_A : M_A \to \{P, O\}$ is the labeling function that assigns who is the next player to move;
>
> $M_A^\star$: is the set of all alternately-labeled finite sequences of moves. Extending this notion, Abramsky defines P_A as the set of positions in a game, which is a non-empty prefix closed subset of $M_A^\star$
>
> W_A is the set of all infinite sequences of moves, all of whose finite prefixes are in P_A. It indicates which of the finite games are won by P.

A strategy for P (which always has its opening move by O) in A is defined as a partial function from positions (with P to move) to moves (executed by P). Abramsky used three conditions in order to define a strategy, which is a non-empty prefix-closed subset σ of P_a

1. The opponent always starts a game, or formally, $a.s \in \sigma \to (a) = O$;
2. Strategies are deterministic, or, if $s.a$, $s.b \in \sigma$, P moving at s, then $a = b$;
3. If $s \in \sigma$, O to move at s, $s.a \in P_a$ then $s.a \in \sigma$.

Conditions 2 and 3 can define a strategy for O (with an opening move by O) in A. To achieve this strategy one has to exchange the roles of opponent and proponent in both conditions and, in doing so, the resulting strategy is called *counter-strategy*. A strategy is a winning strategy if it beats every counter-strategy.

Table 4 is extracted from Abramsky's paper and has been pictured here to facilitate the comprehension of the concepts listed so far. Show this picture is very important in attaining further conclusions and considerations about this game-theoretical approach. Even though this work aims to analyse the approach as a whole rather than its quality in modeling linear logic the following subsection covers the most distinctive characteristic between them – the tensor product's game definition, which actually affects not only the tensor product but all the multiplicative fragment since every connective in that fragment can be determined in terms of tensor product and linear negation.

3.2 Multiplicative's behaviour

In defining how does *linear negation* and *tensor product* work, one attains every connective of linear logic's multiplicative fragment. In the case of linear negation, one can note that it is (apart from dissimilarities on the game's structure), very similar to the definition given by Blass.

Abramsky, then, gives the following interpretation to linear negation. A *linear negation* to a game A is the following structure:

$$A^{\perp} = (M_A, \bar{\lambda}_A, P_A, P_A^{\infty}\ W_A)$$

Further, $\bar{\lambda}_A(a) \equiv \lambda_{\bar{a}}(a)$, yielding the equality $A \equiv A^{\perp\perp}$ (this is one of the main-stones on linear logic – the existence of a constructive and involutive negation).

Once linear negation has been defined, by defining the tensor product's behaviour, one can have the entire set of multiplicatives defined by "De Morgan's Law". A game for tensor product of A and B is defined as follows:

$$A \otimes B = (M_{A\otimes B}, \lambda_{A\otimes B}, P_{A\otimes B}, P_{A\otimes B}^{\infty}\ W_{A\otimes B})$$

Where $M_{A\otimes B}$ is the disjoint union of A and B, $\lambda_{A\otimes B}$ is the labeling function to both games. The component $P_{A\otimes B}$ is the set of alternately-labeled finite sequences of moves such that:

1. The restriction of moves in M_A (or M_B) is in P_A (or in P_B);
2. If two successive moves are in different components, the responsible for this change was the opponent.

Further, $W_{A\otimes B}$ is the set of all infinite plays of the game such that the restriction P_i ($i = A$ or B) is either finite or it is a win for P in that component.

For sake of completeness of multiplicative fragment (and semantics as a whole), it is also defined the unity for tensor product as the following:

$$1 = (\emptyset, \emptyset, \epsilon, \emptyset)$$

And a convention sets that $\perp = 1^{\perp} = 1$ which can be observed when operating with the structure representing a game.

Now one is able to construct a similar interpretation for every connective belonging to the multiplicative fragment using the following rules:

1. $A \widetilde{\otimes} B \leftrightarrow (A^{\perp} \otimes B^{\perp})^{\perp}$
2. $A \multimap B \leftrightarrow (A^{\perp} \widetilde{\otimes} B)$

The subsequent definitions are straightforward but required in order to understand the whole framework. Firstly $A \widetilde{\otimes} B$ follows the same definition of $A \otimes B$ game, except that the proponent is the responsible for switching the components (rather than O, in tensor's definition) and $W_{A \widetilde{\otimes} B}$ is the set of all infinite plays whose restriction to one component or the other is an infinite play. Further, by equivalence 2, s is in $W_{A \perp\circ B}$ just if $s \setminus A \in W_A$ implies that $s \setminus B \in W_B$

where $s \setminus A$ (or B) stands for the resulting sequence s attained by erasing every occurrence of symbols not in A (or B) in that sequence.

The first condition for $A \otimes B$ says that a play in $A \otimes B$ consists of a current play in A or B (note that Blass does not mention it so explicitly but one can feel this underlying idea in his approach). The second condition shows the difference between tensor product and par connective – tensor product represents *disjoint concurrency* while par stands for *connected concurrency*[4]. This means that tensor combines two parallel processes with no flow of information between them while par requires these channels of communication. In a game-theoretical standpoint the considerations above lead to the following reason: the proponent for tensor (opponent for par) is allowed to use moves of his rival in one component to influence his play in the other component.

This strategy is called *copy-cat strategy* and it determines that one player can win a game by playing in one component, exactly the opposite of what was played in the other component (note the connection between this strategy and $\vdash A, A^{\perp}$ axiom, which can naturally be interpreted as $\vdash A \widetilde{\otimes} A^{\perp}$).

Abramsky also gave an interpretation in terms of proof-net however it falls outside the range of this work, which is focused on games rather than strictly linear logic. Yet, the following is very important because this is one result that cannot be achieved in Blass' definition however is natural in Abramsky's one.

3.3 Category of games

Abramsky built a category of games in which objects are games and winning strategies are morphisms. These morphisms $\sigma : A \rightarrow B$ are winning strategies in $A \multimap B$. The first observation that has to be made is that these morphisms, or winning strategies, can be composed in such a way that if $\sigma_1 : A \rightarrow B$ and $\sigma_2 : B \rightarrow C$, one can compose σ_1 and σ_2 so that $\sigma_1;\sigma_2 : A \rightarrow C$. The following defines it in a proper way.

If $\sigma_1 : A \rightarrow B$ and $\sigma_2 : B \rightarrow C$ then:

$$\sigma_1;\sigma_2 = \{s \setminus A, C \mid s \in (A, B, C), s \setminus A, B \in \sigma_1, s \setminus B, C \in \sigma_2\}$$

Where $\mathcal{L}$ stands for the set of every sequence in every game if the component A, B or C. Abramsky points out that this definition exhibits the *cut = parallel composition + hiding* approach as the correct interpretation of "cut" in classical linear logic , with respect to CSP trace semantics style [Hoa85]. Furthermore, the game-theoretical approach reveals exactly who is the opponent (in this case the environment) while trace semantics does not define it clearly.

To sum up with this section is worth noting that $\mathcal{G}$ is in fact a category because besides compositionality of strategies, one has to have the *identity morphism*:

$$\begin{aligned} id_a = \{ & s \in P_{A \perp \circ A} / s \text{ begins with an O-move and for all subsequences } t \text{ of } s, \\ & \text{if } t\text{'s cardinality is even then } t \setminus A = t \setminus A^{\perp} \} \end{aligned}$$

Note that the definition above is strongly connected with the copy-cat strategy idea (the same number in A and $A^{\perp}$).

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[4] For a better understanding use table 4

Moreover, its composition must be associative then $\sigma_1 : A \to B$, $\sigma_2 : B \to C$ and $\sigma_3 : C \to D$, $(\sigma_1;\sigma_2);\sigma_3 = S$ where:

$$S = \{t \setminus A, D \mid t \in \mathcal{L}(A, B, C), t \setminus A, B \in \sigma_1, t \setminus B, C \in \sigma_2, t \setminus C, D \in \sigma_3\}$$

So that the same holds for $\sigma_1;(\sigma_2;\sigma_3)$

So Abramsky's theory has reached a position where one is able to define *full completeness* to this semantics given the categorical approach previously achieved. Again, to explore every aspect surrounding this definition does not lie within the scope of this work so that it is sufficient to show its main result, i.e. the full completeness theorem.

3.4 Full completeness

The most well-know (and used) notion of completeness defines that if some formula is semantically valid in a certain logic, it implies that this formula should be syntactically derivable in the deduction system of this logic. Abramsky, amongst others, uses a stronger concept of completeness, based on the categorical model of a given logic. The underlying idea is that formulas denote objects and proofs denote morphisms, so that one has the following

- Completeness: $\mathbf{C}(A, B)$ is non-empty if B is syntactically derivable from A in a certain logic;
- Full Completeness: any morphism $f : A \to B$ is denotation of a proof that B syntactically from A.

Quoting Abramsky, "*with full completeness one has the tightest possible connection between syntax and semantics*" and this can indeed be noted by the fact that in the syntactical framework intended by Abramsky, the following theorem is established:

Theorem 2 (Full Completeness) *If σ is an uniform, history-free winning strategy for ?, then it is a denotation of a unique proof-net* (? , $\emptyset$)

It remains to explain what does *uniform* and *history-free* strategy is indeed. This task is executed below however, once one has established full completeness theorem for a logic, a stronger result is attained (in comparison with regular completeness). Moreover, it follows that it has every characteristic adjoined by his categorical aspect such as compositionality (which has to be associative), identity (both of them regarding strategies) and dual proof scheme.

An explanation of the terminology introduced below is required and so the subsequent text does it. *Uniform* is a term connected with families of strategies that can be roughly explained as "*whenever there are equivalent history-free strategies, there is an element which is called* **canonical** *such that every morphism (strategy) has this object as image*". This is the concept of "uniform". The other term used in the theorem is "history-free". Abramsky calls a strategy a history-free strategy when this strategy stipulates the proponent's next move is a function defined in terms of opponent's last move, i.e. no previous move rather than the last one is taken into account.

Although this is just a short and rough sketch of the results attained, it must be suffice give a general idea about the approaches described as well as enable the reader to draw himself a concluision based on a comparison of them. The following section presents the author's personal conclusions about these approaches.

4 Discussion

There are two ways to analyse the approaches presented in the previous sections – one can either judge these theories by considering its abilities to deal with linear logic or by stressing their underlying idea and structure. This work uses the latter as a pattern to measure the previously described works, i.e. catching up their ideas and analysing the used to define them.

First of all it is interesting to notice that Abramsky has founded all his work in [Abr93] without referring Lorenzen's work at any time, however one can easily perceive Lorenzen's ideas behind his definition of games.

Before getting into details about these issues, is worthwhile pointing out the main difference between these approaches – the tensor product. Blass considered games that may have as opening moves made either by P or by O (this resembles Abramsky's definition of *positive* and *negative* games respectively; moreover this is related with Girard's concept of polarity even though Abramsky has a slightly different interpretation, which is similar to [Pat92]). This implies that the games itself will be either positive or negative (in the sense described above). Eventually, plays are positive in Blass if they are started by P and negative otherwise.

So Blass defined that P is the player to move next if he is to move in either component. One can easily note that this definition is similar to Abramsky's however one can conduct a debate in such a way that P is to move in both components, what contradicts to the previous assertion that he can move in either component. The thing to be noted here is that following Abramsky's rules it is not possible to attain such situation.

To skirt this problem Blass treats it as a special case where player makes his opening move in both components so that, as pointed out in Abramsky, it opens the range to what he entitle as *Blass pathologies.* Most of the results attained by Abramsky are founded in such issue and the following text, as explained above, makes it clear.

The following subsections attack some issues that came-up when analysing the described approaches. Some insights are completely personal and can be wrong but nevertheless they expose author's personal idea about the games as a kind of semantics.

4.1 Weakening

Following the set of rules established by Blass, the weakening rule is valid. The sequent below represents a standard application of this rule.

$$\frac{\vdash \Delta}{\vdash \Delta, A}$$

It is not difficult to understand why is happens and to do so suppose that proponent has a winning strategy for Δ. Consider the game Δ, A and suppose that A is a game for which proponent provides the opening moves. Then O cannot move in A since P is the player who can switch components in *par*. So one does not ever need to play in A.

Now, suposse that A is a game such that every opening move is executed by O. Two situations can be reached in considering the previous request.

- There is a game in Δ which P can make his opening move so he is to start in Δ and Δ, A. By playing just in Δ and never getting into A.
- There is no game in Δ, A such that P makes the opening. Then the situation described on the preamble of this section takes place and O has to make his opening move in every component of Δ, A. Then P can restrict himself in playing just in Δ.

Clearly, the above options turn out to validate weakening since, given a strategy for Δ, on is able to drawn Δ, A as a valid conclusion. Conversely however, following the Abramsky's set of rules, except when A is a game for which every opening move is by P, O is able to move in A, and P is bound to answer. So weakening is no longer valid.

4.2 Composition of Strategies

Taking into account the approach proposed in Blass, if one tries to compose strategies, the result is that they would not be associative. One of the characteristics of the categorical approach is that the operation should be associative so that Blass is unable to come-up with such category. On the other hand, considering Abramsky's strategy, one can see that they can be composed and associative, therefore Abramsky really gets a category of games.

The following example creates a suitable environment where one can understand why this happens. The example is not due to the author but instead was developed in [Abr93]. It is worth explaining that it uses the special case mentioned in the preamble of these section and also uses the mathematics briefly explained above. The author tried, inasmuch as possible, to accomplish the task of giving the most clear explanation possible in order to make this example sound and intuitive.

The first step consists in defining four games that are called respectively, A, B, C and D. They are formally defined as follows:[5]

$$\begin{aligned}
A &= (\{a_1, a_2\}, \{(a_1, P), (a_2, O)\}, \{(a_1, a_2)^\star \cdot (\epsilon + a_1)\}, \emptyset\}) \\
B &= (\{b_1, b_2\}, \{(b_1, O), (b_2, P)\}, \{(b_1, b_2)^\star \cdot (\epsilon + b_1)\}, \emptyset\}) \\
C &= (\{c_1, c_2\}, \{(c_1, P), (c_2, O)\}, \{(c_1, c_2)^\star \cdot (\epsilon + c_1)\}, \{c_1 \cdot c_2\}^\omega\})
\end{aligned}$$

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[5] The game structure taken into account in this stage regards the standard game used in Abramsky

$$D \;=\; (\{d_1, d_2\}, \{(d_1, O), (d_2, P)\}, \{(d_1, d_2)^\star \cdot (\epsilon + d_1)\}, \{d_1 \cdot d_2\}^\omega\})$$

To refresh the reader's memory, the first component stands for the set of possible moves, the second for the labelling function, the third component ranges over the sequence of moves and finally, the fourth component is the set of infinite sequences won by P. Now, there are three winning strategies assumed in Abramsky, namely σ, τ and υ and they are mathematically defined as follows:

$$\sigma = \{(\langle a_1, b_1\rangle, a_2)\} \cup \{(s, a_2) | s \in \langle a_1, b_1\rangle \cdot (a_2 \cdot a_1)^\star\}$$

$$\tau = \{\epsilon, \langle b_1, c_1\rangle)\} \cup \{(s, b_1) | s \in \langle b_1, c_1\rangle \cdot b_2 \cdot (b_1 \cdot b_2)^\star\} \cup \{(s, c_1) | s \in \langle b_1, c_1\rangle \cdot c_2 \cdot (c_1 \cdot c_2)^\star\}$$

$$\upsilon = \{(\langle c_1, d_1\rangle, d_2)\} \cup \{(s, d_2) | s \in \langle c_1, d_1\rangle \cdot (d_2 \cdot d_1)^\star\}$$

Consider the move $\langle a, b\rangle$ as a move that has the special case type. The following explains the intuition behind those strategies.

- σ is the strategy that makes the entire play to be played in $A^\perp$;
- τ is the strategy that forces the entire play to be played in the component chosen by O in response to P's first move;
- υ is the strategy that forces the game to be played in the component D

Now it one is able to consider the strategy $\sigma; \tau : A \to B$ where ';' stands for the composition of the two strategies. So O starts in C. The answer to this move happens in C, which P uses the strategy τ. The resulting strategy never makes use of any move in τ. This corresponds to the following set of moves:

$$\sigma; \tau = \{(s, c_1) | s \in a_1 \cdot (c_1 \cdot c_2)^\star\}$$

In following the same pattern, one is able to analyse the composition $\tau; \upsilon : B \to D$. In this composition, O makes the its initial move in D because υ cannot be used. Then P replies with a move in B and the game keeps on this component so that the υ is never used. Mathematically, it is equivalent to the subsequent set:

$$\tau; \upsilon = \{(s, b_1) | s \in d_1 \cdot (b_1 \cdot b_2)^\star\}$$

By enlarging the proposed procedure and following the same method of construction applied above one can compose three strategies. Now the crucial issue is brought to light — the special case raises the issue that Abramsky had on mind when posing the example, namely, in composing these three strategies one finds different results so, compositionality is not associative. The following shows that in terms of sets:

Suppose the following compositions $(\sigma; \tau); \upsilon : A \to D$ and also $\sigma; (\tau; \upsilon) : A \to D$. Using the mathematical procedure employed above one reachs the subsequent results:

$$(\sigma; \tau); \upsilon = \{(\langle a_1, d_1\rangle, d_2)\} \cup \{(s, d_2) | s \in (\langle a_1, d_1\rangle \cdot (d_2 \cdot d_1)^\star\}$$

$$\sigma;(\tau;\upsilon) = \{(\langle a_1, d_1\rangle, a_2)\} \cup \{(s, a_2) | s \in (\langle a_1, d_1\rangle \cdot (a_2 \cdot a_1)^\star\}$$

And it is clear that these strategies are not equivalent since in the former the resulting strategy leads to a game which is played in the component D whereas the latter sets the game in $A^\perp$.

4.3 Further discussion

Van Benthem [Ben98] came-up with an interesting question: is it possible to model *resource consciouness* in terms of attacks and defense conventions? The approaches described in section 2 and 3 considered infinite games as a method to model linear logic, however it does not make too much sense to think about it as infinite since the fact being considered is finiteness of resources. Even though Abramsky's process interpretation is suitable — a valid formula stands for a deadlock free process.

The following texts attemps to show a intuitive glance of ideas concerning a sketch of model that claims to validate multiplicative linear logic. One has to understand that this is indeed a big goal that cannot, in any way, be attained and proved in these rough subsection. However it is hoped that further standpoints and criticism may raise from this and an improvement may be achieved.

From now on both Blass, Abramsky and Lorenzen approaches are disregarded. Therefore the idea considered here includes players as used in those approaches, games as finite chattering between P and O and rules and objectives to each player – as usual P has to show that some formula is valid while O has the opposite task.

Therefore firstly one has to define the rules that regulate players' behaviour in a debate. The set of rules below was drained from practical experiences that were made in order to understand how does a debate should be conducted.

1. If some player asserts a formula A, he cannot assert its dual $A^\perp$;
2. O is the starting player for tensor;
3. Neither P nor O can re-attack or re-defend any formula (whether this formula is atomic or not);
4. A (atomic) formula can be asserted at most once.

Rule 1 is adjoined to control consistency. Here the concept give in [Gir87] is brought to light — if one allows either player to assert a formula and its negation, a invalid trip is reached (in terms of *proof-net*). It is also possible to understand it as another concept (also considered in [Gir93]) that is related with *input* and *output* as an suitable interpretation for a formula and its dual respectively.

Rule 2 concerns the starting player for a tensor product game. Considering O as the starting player mimics the intuition that the onus in showing that a certain statement is false is from opponent. This idea is clear in [Bla92] and has came-up naturally in practical games.

Rules 3 and 4 are the *kernel* of the resource consciousness. In not allowing players to re-attack and re-defend (and stipulating that a formula may be *used* just once) is essential for achieving the proposed idea. One is able to see that these rules can be colapsed into a single rule but for sake of clarity they were left apart.

Once the underlying rules were presented, the forthcoming step is to define how does tensor product (*par*) and linear negation look like. Table 5 table does this task.

Linear implication ($\multimap$) is defined as the equivalence 2 on page 9 and linear negation is just the roles' change. In attaining such rules, some examples were raised and the table 6 shows that *contraction* and *weakening* are both not valid.

Tensor	**Par**
Assertion: P asserts $\psi \otimes \gamma$	**Assertion**: P asserts $\psi \widetilde{\otimes} \gamma$
Attack: O attacks the whole $\otimes$	**Attack**: P points out and disjunct
Defense: P has to show assert both conjunctor in a row	**Defense**: O asserts its negation

Table 5: Proposed description of multiplicative connectives behaviour.

Weakening $A \multimap (A \otimes A)$	**Contraction** $(A \otimes A) \multimap A$
Equivalent game: $A^{\perp} \widetilde{\otimes} (A \otimes A)$	**Equivalent game**: $(A \otimes A)^{\perp} \widetilde{\otimes} A$
P asserts $A^{\perp}$	P asserts A
O pick the conjunction	O asserts $\otimes$
P cannot play A – O wins	P cannot pick $A^{\perp}$ – O wins

Table 6: Example of games.

In order to show how it works, a example for tensor and for linear implication are given in the following text. Suppose that

$$\frac{?\,, A \qquad\qquad B, \Delta}{?\,, A \otimes B, \Delta}$$

And suppose that τ and σ are winning strategies. Following the conventions above, P asserts $\otimes$, O attacks the tensor product and P has to show $A \otimes B$. From this, two option may araise:

- → If P starts in $?\,, A$, apply τ;
- → If P starts in B, Δ, apply σ.

Then, either situation can happen:

→ τ wins in ?, or A or;

→ σ wins in Δ, or A.

So the strategy $\tau \otimes \sigma$ is sound since the situations explained above are winning strategies in $A \otimes B$

Another example that can be shown here is the linear implication connective. Clearly the following cannot be valid:

$$A^{\perp} \multimap A$$

this can be transformed in a par game by the equivalences given in page 9 and following this game using the stipulated set of rules, O wins.

Similarly *axiom-link* and *cut rule* are valid considering the rules above.

Surely this exposition is weak since it lacks of formal proof as well as a formal schemata. Paper's author prefers to avoid these machinery at this stage since they require more than this simple set of rules. Nevertheless some questions may be raised at this point:

> What happen to the additives? And with the exponentials?
>
> Can one think of compositionality of strategies?

These questions were raised after the definition above and still not answered. One may use [Lor59] or [Fel86] as game definition to the additives so that the above system is valid just for MLL. In exponenctial's case one can consider them as a series of tensor (and dually, par) games but it has not been testified yet.

To the latter question, the lack of proof and formal background does not allow any further conclusion but the author belives in its possibility.

Regarding soundness of the proposed system, one can verify that it is possible to determine games for *par* and *linear implication* using the *De Morgan* correspondencies [6]. So, once the game structure is correctly defined for the tensor, a naive implication leads to the belief that this approach is correct and sound. However a deeper analysis has to be made and the mathematical background involved in such a structure has to be defined in order so that a concrete proof could be established.

Still, the claim that this model represents the promissed intention, i.e., to model (in finite games) a system based on linear logic is rather intuitive however the author believes in its correcteness for the exploited fragment and also that the set of rules and restrictions suffices to determine it. Finite games would stand for finiteness of resources in such a way that a correspondence between formula occurrences and possible arguments is set. Therefore, a game that has the entire set of *arguments* used, stands for a process (disregarding who was the winner in such game) such that every resource was spend. Conversely, in finishing a game where one finds *leftovers*, the correspondence represents a misuse of resources.

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[6] C.f. page 9

A Rules for Classical Linear Logic

$$\frac{}{\vdash A, A^{\perp}}\ \mathit{Identity} \qquad \frac{\vdash \Gamma, A \vdash A\perp, \Delta}{\vdash \Gamma, \Delta}\ \mathit{Cut\perp rule} \qquad \frac{\vdash \Gamma}{\vdash \Gamma'}\ \mathit{Exchange}$$

$$\frac{}{\vdash 1}\ \mathit{One} \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \mathit{False}$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \mathit{Tensor} \qquad\qquad \frac{\vdash \Gamma, A, B}{\Gamma, A \widetilde{\otimes} B}\ \mathit{Par}$$

$$\frac{}{\vdash \Gamma, \top}\ \mathit{True}$$

No rule for O

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A\ B, \Delta}\&\mathit{With} \qquad\qquad \frac{\vdash \Gamma, A}{\Gamma, A \oplus B}\ \mathit{Plus}$$

$$\frac{\vdash \Gamma\Gamma, A}{\vdash \Gamma\Gamma, !A}\ \mathit{Ofcourse} \qquad\qquad \frac{\vdash \Gamma}{\vdash \Gamma, \Gamma A}\ \mathit{Weakening}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \Gamma A}\ \mathit{Dereliction} \qquad\qquad \frac{\vdash \Gamma, \Gamma A, \Gamma A}{\vdash \Gamma, \Gamma A}\ \mathit{Contraction}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall\xi(A)}\ \mathit{Univ.\ Quant.} \qquad\qquad \frac{\vdash \Gamma, A[t/\xi]}{\vdash \Gamma, \exists\xi.A}\ \mathit{Exist.\ Quant.}$$

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